HOMOLOGY OVER LOCAL HOMOMORPHISMS

By LUCHEZAR L. AVRAMOV, SRIKANTH IYENGAR, and CLAUDIA MILLER

Abstract. The notions of Betti numbers and of Bass numbers of a finite module N over a local ring R are extended to modules that are only assumed to be finite over S, for some local homomorphism $\varphi\colon R \to S$. Various techniques are developed to study the new invariants and to establish their basic properties. In some cases they are computed in closed form. Applications go in several directions. One is to identify new classes of finite R-modules whose classical Betti numbers or Bass numbers have extremal growth. Another is to transfer ring theoretical properties between R and S in situations where S may have infinite flat dimension over R. A third is to obtain criteria for a ring equipped with a "contracting" endomorphism—such as the Frobenius endomorphism—to be regular or complete intersection; these results represent broad generalizations of Kunz's characterization of regularity in prime characteristic.

Introduction. The existence of a homomorphism $\varphi\colon R \to S$ of commutative noetherian rings does not imply a relationship between ring theoretical properties of R and S, such as regularity, normality, Cohen-Macaulayness, etc. It is therefore remarkable that certain homological conditions on the *R-module S* force stringent relations between the *ring* structures of R and S. A classical chapter of commutative algebra, started by Grothendieck, deals with the case when S is *flat* over R. Parts of this theory have been extended to a situation where S is only assumed to have finite flat dimension over R.

An initial motivation for this investigation was to find conditions on the R-module S that allow a transfer of properties between the rings even in cases of infinite flat dimension. It became rapidly apparent that such a program requires new invariants. Our first objective is to introduce homological measures for finite S-modules, which reflect their structure as R-modules. In the special case when φ is the identity map of R, they reduce to classical invariants of finite R-modules. In general, they have properties that adequately extend those of their counterparts in the finite case. Our main goal is to demonstrate the usefulness of the new concepts for studying homomorphisms of commutative noetherian rings. The central case is when the homomorphism φ is local, which means that the rings R and S are local and φ maps the unique maximal ideal of R into that of S.

We start by constructing sequences of invariants modeled on the sequences of integers, the Betti numbers and the Bass numbers, classically attached to a finite R-module M. One way to introduce them is as ranks of modules in minimal

Manuscript received December 23, 2003.
Research supported in part by grants from the National Science Foundation.
American Journal of Mathematics 128 (2006), 23–90.

resolutions. Another is as ranks of the vector spaces $\mathrm{Tor}_n^R(k,M)$ or $\mathrm{Ext}_R^n(k,M)$, where k is the residue field of R. However, for a finite S-module N neither approach provides finite numbers in general.

We define Betti numbers $\beta_n^{\varphi}(N)$ and Bass numbers $\mu_{\varphi}^n(N)$ of N over φ, in a way that ensures their finiteness when N is finite over S. To this end we use the fact that $\mathrm{Tor}_n^R(k,N)$ or $\mathrm{Ext}_R^n(k,N)$ have natural structures of finite S-modules, and that this holds even when N is a homologically finite complex of S-modules. This is the contents of Section 4. The necessary machinery is assembled in the first three sections of the paper. It is put to different use in Section 2 where it is applied in conjunction with the "Bass conjecture" to prove that if a finite S-module has finite injective dimension over R, then R is Cohen-Macaulay.

Under special conditions we compute in closed form the entire sequence of Betti numbers or of Bass numbers. Results are often best stated in terms of the corresponding generating function, called the Poincaré series or the Bass series of N over φ, respectively. Section 5 contains instances of such computations, intended both as illustration and for use later in the paper.

We start Section 6 by establishing upper bounds for the Poincaré series and the Bass series of N over φ, in terms of expressions where the contributions of R, S, and φ appear as separate factors. When the bound for Poincaré series is reached the module N is said to be separated over φ; it is said to be injectively separated over φ when the Bass series reaches its bound. We study such modules in significant detail. Using numerical invariants of the S-module N obtained from a Koszul complex and analyzed in Section 3, we prove that the Betti numbers of separated modules share many properties with the classical Betti numbers of k over R. It came as a surprise (to us) that separated modules occur with high frequency. For example, when the ring S is regular every S-module is separated over φ.

As we do not assume the homological dimensions of N over R to be finite, the sequences of Betti numbers and of Bass numbers of N over φ may contain a lot of inessential information. Some of our main results show that the asymptotic behavior of these sequences captures important aspects of the structure of N. Comparisons of Betti sequences to polynomial functions and to exponential functions lead us in Section 7 to the notions of complexity $\mathrm{cx}_{\varphi} N$ and curvature $\mathrm{curv}_{\varphi} N$, respectively: Injective invariants are similarly derived from Bass numbers.

The next four sections are devoted to the study of various aspects of these new invariants. In Section 8 we analyze their dependence on S, and prove that this ring can be replaced by any other ring S' over which N is finite module, provided its action is compatible with that of S. In particular, it follows that when the R-module N is finite its complexities and curvatures over φ are equal to those over R, although the corresponding Betti numbers or Bass numbers may differ substantially. In Section 9 we investigate how (injective) complexities and curvatures change under compositions of homomorphisms. In Section 10 we prove that they do not go up under localization. In Section 11 we give conditions that ensure that the asymptotic invariants of N over φ have the maximal possible values, namely, those of the corresponding invariants of k over R.

In Section 12 we focus on the case when φ is a contracting endomorphism of R, by which we mean that for every non-unit $x \in R$ the sequence $(\varphi^i(x))$ converges to 0 in the natural topology of R. The motivating example is the Frobenius endomorphism, but interesting contractions exist in all characteristics.

The final Section 13 contains applications of the methods developed in the paper to the study of local homomorphisms. We obtain results on the descent of certain ring theoretical properties from S to R. We also prove that when φ is a contraction its homological properties determine whether R is regular or complete intersection. Thus, we obtain vast generalizations and a completely new proof of Kunz's famous characterization of regularity in prime characteristic. Some of our results concerning the Frobenius endomorphism are announced in Miller's survey [28] of the homological properties of that map.

Even when dealing with modules, in some constructions and in many proofs we use complexes. We have therefore chosen to develop the entire theory in terms of the derived categories of R and S, stating both definitions and results for homologically finite complexes of modules over S, rather than just for finite S-modules.

Notation. Throughout this paper $(R, \mathfrak{m}, k)$ denotes a *local ring*: this means (for us) that R is a commutative noetherian ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We also fix a *local homomorphism*

$$\varphi\colon\ (R, \mathfrak{m}, k) \longrightarrow (S, \mathfrak{n}, l)$$

that is, a homomorphism of noetherian local rings such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

As usual, $\operatorname{edim} S$ stands for the *embedding dimension* of S, defined as the minimal number of generators of its maximal ideal $\mathfrak{n}$. In addition, we set

$$\operatorname{edim} \varphi = \operatorname{edim} (S/\mathfrak{m}S)$$

A *set of generators of* $\mathfrak{n}$ *modulo* $\mathfrak{m}S$ is a finite subset $\mathbf{x}$ of $\mathfrak{n}$ whose image in $S/\mathfrak{m}S$ generates the ideal $\mathfrak{n}/\mathfrak{m}S$. Such a set is *minimal* if no proper subset of $\mathbf{x}$ generates $\mathfrak{n}$ modulo $\mathfrak{m}S$; by Nakayama's Lemma, this happens if and only if $\operatorname{card}(\mathbf{x}) = \operatorname{edim} \varphi$.

Throughout the paper, N denotes a complex of S-modules

$$\cdots \longrightarrow N_{n+1} \xrightarrow{\partial^N_{n+1}} N_n \xrightarrow{\partial^N_n} N_{n-1} \longrightarrow \cdots$$

which is *homologically finite*, that is, the S-module $\mathrm{H}_n(N)$ is finite for each n and vanishes for almost all $n \in \mathbb{Z}$.

Complexes of S-modules are viewed as complexes of R-modules via φ. Modules are identified with complexes concentrated in degree 0.

We let $\widehat{S}$ denote the $\mathfrak{n}$-adic completion of S and set $\widehat{N} = N \otimes_S \widehat{S}$. This complex is homologically finite over $\widehat{S}$: the flatness of $\widehat{S}$ over S yields $\mathrm{H}_n(\widehat{N}) \cong \mathrm{H}_n(N) \otimes_S \widehat{S}$.

Acknowledgments. Part of the work on this paper was done during the Spring semester of 2003 at MSRI, where the authors participated in the Program in Commutative Algebra, and during visits of C.M. (August 2002–December 2002) and S.I. (August 2003–May 2004) to the University of Nebraska, Lincoln. The hospitality of both institutions is gratefully acknowledged.

Part of the work on this paper was also done while one of us was collaborating with Sean Sather-Wagstaff on [20], where the focus is on finite Gorenstein dimension. We thank Sean for many useful discussions, which have been beneficial to this project, and for a very thorough reading of its outcome. Thanks are also due to Anders Frankild for raising the question of characterizing regularity in terms of the injective dimension of the Frobenius endomorphism, settled in Theorem 13.3, and to the referee for useful suggestions on the exposition.

1. Complexes. This article deals mainly with homologically finite complexes. However, it is often convenient, and sometimes necessary, to operate in the full derived category of complexes.

1.1. Derived categories. For each complex M, we set

$$\sup M = \sup\{n \in \mathbb{Z} \mid M_n \neq 0\} \quad \text{and} \quad \inf M = \inf\{n \in \mathbb{Z} \mid M_n \neq 0\}.$$

We say that M is *bounded* when both numbers are finite and that it is *homologically bounded* when $\mathrm{H}(M)$ is bounded.

Let $\mathcal{D}(R)$ denote the derived category of complexes of R-modules, obtained from the homotopy category of complexes of R-modules by localizing at the class of homology isomorphisms. The procedure for constructing this localization is the same as for derived categories of bounded complexes, see Verdier [37], once appropriate "resolutions" are provided, such as the K-resolutions of Spaltenstein [34].

The symbol $\simeq$ denotes isomorphism in a derived category, and Σ the shift functor. We identify the category of R-modules with the full subcategory of $\mathcal{D}(R)$ consisting of the complexes with homology concentrated in degree zero, and let $\mathcal{D}^{\mathrm{f}}(R)$ denote the full subcategory of homologically finite complexes.

The derived functors of tensor products and of homomorphisms are denoted $(- \otimes^{\mathbf{L}}_R -)$ and $\mathbf{R}\mathrm{Hom}_R(-,-)$, respectively. These may be defined as follows: for complexes of R-modules X and Y, let P be a K-projective resolution of X, and set

$$X \otimes^{\mathbf{L}}_R Y \simeq P \otimes_R Y \qquad \text{and} \qquad \mathbf{R}\mathrm{Hom}_R(X, Y) \simeq \mathrm{Hom}_R(P, Y).$$

In particular, when Y is a complex of S-modules, so are $P \otimes_R Y$ and $\mathrm{Hom}_R(P, Y)$. In this way, $- \otimes^{\mathbf{L}}_R N$ and $\mathbf{R}\mathrm{Hom}_R(-, N)$ define functors from $\mathcal{D}(R)$ to $\mathcal{D}(S)$.

For each integer n we set

$$\mathrm{Tor}^R_n(X, Y) = \mathrm{H}_n(X \otimes^{\mathbf{L}}_R Y) \qquad \text{and} \qquad \mathrm{Ext}^n_R(X, Y) = \mathrm{H}_{-n}(\mathbf{R}\mathrm{Hom}_R(X, Y)).$$

When Y is a complex of S-modules, $\mathrm{Tor}^R_n(X, Y)$ and $\mathrm{Ext}^n_R(X, Y)$ inherit S-module structures from $P \otimes_R Y$ and $\mathrm{Hom}_R(P, Y)$, respectively.

The rest of the section is a collection of basic tools used frequently in the paper.

1.2. Endomorphisms. Let X be a complex of S-modules. Let $\theta\colon X \to X$ be a morphism in $\mathcal{D}(S)$ and let $X \xrightarrow{\theta} X \to C \to$ be a triangle in $\mathcal{D}(S)$.

The homology long exact sequence yields

1.2.1. For each integer n, there exists an exact sequence of S-modules

$$0 \longrightarrow \mathrm{Coker}\,\mathrm{H}_n(\theta) \longrightarrow \mathrm{H}_n(C) \longrightarrow \mathrm{Ker}\,\mathrm{H}_{n-1}(\theta) \longrightarrow 0.$$

In particular, since S is noetherian, if $\mathrm{H}_n(X)$ is finite for each $n \in \mathbb{Z}$ (respectively, if X is homologically finite), then C has the corresponding property.

The next statement is a slight extension of [17, (1.3)].

1.2.2. The following inequalities hold:

$$\sup \mathrm{H}(C) \leq \sup \mathrm{H}(X) + 1 \qquad \text{and} \qquad \inf \mathrm{H}(X) \leq \inf \mathrm{H}(C).$$

In addition, if $\mathrm{Im}\,\mathrm{H}(\theta) \subseteq \mathfrak{n}\mathrm{H}(X)$, then

$$\sup \mathrm{H}(X) \leq \sup \mathrm{H}(C) \qquad \text{and} \qquad \inf \mathrm{H}(X) = \inf \mathrm{H}(C).$$

Indeed, the (in)equalities follow from the exact sequences above; under the additional hypothesis, Nakayama's Lemma has to be invoked as well.

The symbol ℓ_S denotes length over S.

LEMMA 1.2.3. *If each $\ell_S\mathrm{H}_n(X)$ is finite, and $\mathrm{H}(\theta)^v = 0$ for some $v \in \mathbb{N}$ then for each $n \in \mathbb{Z}$ there are inequalities*

$$v^{-1}\left(\ell_S\mathrm{H}_n(X) + \ell_S\mathrm{H}_{n-1}(X)\right) \leq \ell_S\mathrm{H}_n(C) \leq \ell_S\mathrm{H}_n(X) + \ell_S\mathrm{H}_{n-1}(X).$$

Proof. Set $\alpha_n = \mathrm{H}_n(\theta)$. A length count on the short exact sequence (1.2.1) yields

$$\ell_S \mathrm{H}_n(C) = \ell_S \operatorname{Coker}(\alpha_n) + \ell_S \operatorname{Ker}(\alpha_{n-1}).$$

Since α is nilpotent of degree v, each S-module $\mathrm{H}_n(X)$ has a filtration

$$0 = \operatorname{Im}(\alpha_n^v) \subseteq \operatorname{Im}(\alpha_n^{v-1}) \subseteq \cdots \subseteq \operatorname{Im}(\alpha_n) \subseteq \operatorname{Im}(\alpha_n^0) = \mathrm{H}_n(X)$$

by S-submodules. As α_n induces for each i an epimorphism

$$\frac{\operatorname{Im}(\alpha_n^{i-1})}{\operatorname{Im}(\alpha_n^{i})} \longrightarrow \frac{\operatorname{Im}(\alpha_n^{i})}{\operatorname{Im}(\alpha_n^{i+1})},$$

we get the inequality on the left hand side in the following formula

$$\frac{1}{v}\ell_S \mathrm{H}_n(X) \leq \ell_S \operatorname{Coker}(\alpha_n) = \ell_S \operatorname{Ker}(\alpha_n) \leq \ell_S \mathrm{H}_n(X).$$

The inequality on the right is obvious. The equality comes from the exact sequence

$$0 \longrightarrow \operatorname{Ker}(\alpha_n) \longrightarrow \mathrm{H}_n(X) \xrightarrow{\alpha_n} \mathrm{H}_n(X) \longrightarrow \operatorname{Coker}(\alpha_n) \longrightarrow 0$$

To get the desired inequalities put together the numerical relations above. □

1.3. Actions. Here is another way to describe the action of S on homology; see (1.1). We let $\lambda_s^X\colon X \to X$ denote multiplication with $s \in S$ on a complex X.

1.3.1. Recall that $\operatorname{Ext}_S^0(N,N)$ is the set of homotopy classes of S-linear morphisms $G \to G$, where G is a projective resolution of N over S. Compositions of morphisms commute with the formation of homotopy classes, so $\operatorname{Ext}_S^0(N,N)$ is a ring. The map assigning to each $s \in S$ the morphism $\lambda_s\colon G \to G$ defines a homomorphism of rings $\eta_S\colon S \to \operatorname{Ext}_S^0(N,N)$ whose image lies in the center of $\operatorname{Ext}_S^0(N,N)$, and

$$\begin{tikzcd}
S \arrow[r, "\eta_S"] & \operatorname{Ext}_S^0(N,N) \arrow[d, "\operatorname{Ext}_\varphi^0(N,N)"] \\
R \arrow[u, "\varphi"] \arrow[r, "\eta_R"] & \operatorname{Ext}_R^0(N,N)
\end{tikzcd}$$

is a commutative diagram of homomorphisms of rings. If σ is any morphism in the homotopy class $\operatorname{Ext}^0_\varphi(N,N)\eta_S(s)$, then the maps $\mathrm{H}(M\otimes^{\mathbf{L}}_R\sigma)$ and $\mathrm{H}\mathbf{R}\mathrm{Hom}_R(M,\sigma)$ coincide with multiplication by s on $\operatorname{Tor}^R(M,N)$ and $\operatorname{Ext}_R(M,N)$, respectively.

LEMMA 1.3.2. *If* $\inf \mathrm{H}(M) > -\infty$ *and each* R*-module* $\mathrm{H}_n(M)$ *is finite, then all* S*-modules* $\operatorname{Tor}^R_n(M,N)$ *and* $\operatorname{Ext}^n_R(M,N)$ *are finite, and are trivial for all* $n \ll 0$.

Proof. Since N is homologically finite over the noetherian ring S, it is isomorphic in the derived category of S-modules to a finite complex of finite S-modules. Changing notation if necessary, we may assume that N itself has these properties. On the other hand, since R is noetherian, one may choose a K-projective resolution P such that for each n, the R-module P_n is finite and projective and $P_n = 0$ for $n < \inf \mathrm{H}(M)$. The complexes $P\otimes_R N$ and $\operatorname{Hom}_R(P,N)$ then consist of finite S-modules, so the desired assertion follows. □

1.4. Annihilators. The annihilator of an S-module H is denoted $\operatorname{Ann}_S(H)$.

The *homotopy annihilator* of N over S, introduced by Apassov [3], is the ideal $\operatorname{Ann}_{\mathcal{D}(S)}(N) = \operatorname{Ker}(\eta_S)$. With $\operatorname{rad}(I)$ denoting the radical of I, the first inclusion below follows from (1.3.1) and the second is [3, Theorem, §2].

1.4.1. There are inclusions $\operatorname{Ann}_{\mathcal{D}(S)}(N)\subseteq \operatorname{Ann}_S(\mathrm{H}(N))\subseteq \operatorname{rad}\big(\operatorname{Ann}_{\mathcal{D}(S)}(N)\big)$.

The first assertion below follows from (1.3.1), the second from (1.4.1).

1.4.2. The ideal $\operatorname{Ann}_{\mathcal{D}(R)}(M)S+\operatorname{Ann}_{\mathcal{D}(S)}(N)$ is contained in the homotopy annihilators of the complexes of S-modules $M\otimes^{\mathbf{L}}_R N$ and $\mathbf{R}\mathrm{Hom}_R(M,N)$, and hence annihilates $\operatorname{Tor}^R_n(M,N)$ and $\operatorname{Ext}^n_R(M,N)$ for each $n\in\mathbb{Z}$.

1.4.3. If $\mathrm{H}(N)/\mathfrak{m}\mathrm{H}(N)$ is artinian over S (in particular, if $\mathrm{H}(N)$ is finite over R, or if the ring $S/\mathfrak{m}S$ is artinian), then the ideal $\mathfrak{m}S+\operatorname{Ann}_{\mathcal{D}(S)}(N)$ is $\mathfrak{n}$-primary.

Indeed, it follows from (1.4.1) that the radical of $\mathfrak{m}S+\operatorname{Ann}_{\mathcal{D}(S)}(N)$ equals the radical of $\mathfrak{m}S+\operatorname{Ann}_S(\mathrm{H}(N))$, which is $\mathfrak{n}$ in view of the hypothesis.

1.5. Koszul complexes. Let $\mathbf{x}$ be a finite set of elements in S. The *Koszul complex* $\mathrm{K}[\mathbf{x};S]$ is the DG (= differential graded) algebra with underlying graded algebra the exterior algebra on a basis $\{e_x\colon\ |e_x|=1\}_{x\in\mathbf{x}}$ and differential given by $\partial(e_x)=x$ for each $x\in\mathbf{x}$. Let $\mathrm{K}[\mathbf{x};N]$ be the DG module $\mathrm{K}[\mathbf{x};S]\otimes_S N$ over $\mathrm{K}[\mathbf{x};S]$.

1.5.1. In the derived category of S-modules there are isomorphisms

$$M\otimes^{\mathbf{L}}_R \mathrm{K}[\mathbf{x};N]\simeq M\otimes^{\mathbf{L}}_R \mathrm{K}[\mathbf{x};S]\otimes^{\mathbf{L}}_S N\simeq \mathrm{K}[\mathbf{x};M\otimes^{\mathbf{L}}_R N]$$
$$\mathbf{R}\mathrm{Hom}_R(M,\mathrm{K}[\mathbf{x};N])\simeq \mathrm{K}[\mathbf{x};\mathbf{R}\mathrm{Hom}_R(M,N)]$$

because $\mathrm{K}[\mathbf{x};S]$ is a finite free complex over S.

1.5.2. There is an isomorphism of complexes of S-modules

$$\mathrm{Hom}_S(\mathrm{K}[\mathbf{x};S],N) \cong \Sigma^{-\operatorname{card}(\mathbf{x})}\mathrm{K}[\mathbf{x};N].$$

Indeed, the canonical morphism of complexes of S-modules

$$\mathrm{Hom}_S(\mathrm{K}[\mathbf{x};S],S)\otimes_S N \to \mathrm{Hom}_S(\mathrm{K}[\mathbf{x};S],N)$$

is bijective, since $\mathrm{K}[\mathbf{x};S]$ is a finite free complex over S, and the self-duality of the exterior algebra yields an isomorphism $\mathrm{Hom}_S(\mathrm{K}[\mathbf{x};S],S) \cong \Sigma^{-\operatorname{card}(\mathbf{x})}\mathrm{K}[\mathbf{x};S]$.

1.5.3. The homotopy annihilator of $\mathrm{K}[\mathbf{x};N]$ contains $\mathbf{x}S + \mathrm{Ann}_{\mathcal{D}(S)}(N)$.

Indeed, the Leibniz rule for the DG module $K = \mathrm{K}[\mathbf{x};N]$ shows that left multiplication with $e_x \in \mathrm{K}[\mathbf{x};S]$ on K is a homotopy between λ_x^K and 0, so $\mathrm{Ann}_{\mathcal{D}(S)}(K)$ contains $\mathbf{x}$. If follows from the definitions that it also contains $\mathrm{Ann}_{\mathcal{D}(S)}(N)$.

It is known that Koszul complexes can be described as iterated mapping cones:

1.5.4. For each $x \in \mathbf{x}$, the Koszul complex $\mathrm{K}[\mathbf{x};N]$ is isomorphic to the mapping cone of the morphism $\lambda_x^{\mathrm{K}[\mathbf{x}';N]}$, where $\mathbf{x}' = \mathbf{x} \smallsetminus \{x\}$.

LEMMA 1.5.5. *The complex of S-modules $\mathrm{K}[\mathbf{x};N]$ is homologically finite, and*

$$\mathrm{Supp}_S \mathrm{H}(\mathrm{K}[\mathbf{x};N]) = \mathrm{Supp}_S(S/\mathbf{x}S) \cap \mathrm{Supp}_S \mathrm{H}(N).$$

Proof. We may assume $\mathbf{x} = \{x\}$. In view of the preceding observation, finiteness follows from (1.2.1). By the same token, one gets the first equality below

$$\begin{aligned}\mathrm{Supp}_S \mathrm{H}(\mathrm{K}[x;N]) &= \mathrm{Supp}_S(\mathrm{H}(N)/x\mathrm{H}(N)) \cup \mathrm{Supp}_S\left(\mathrm{Ker}\,\lambda_x^{\mathrm{H}(N)}\right)\\ &= \mathrm{Supp}_S(S/\mathbf{x}S) \cap \mathrm{Supp}_S \mathrm{H}(N).\end{aligned}$$

To get the second equality, remark that $\mathrm{Supp}_S \mathrm{H}(N) \cap \mathrm{Supp}_S(S/\mathbf{x}S)$ is equal to the support of $\mathrm{H}(N)/x\mathrm{H}(N)$ and contains that of $\mathrm{Ker}\,\lambda_x^{\mathrm{H}(N)}$. □

LEMMA 1.5.6. *For each $n \in \mathbb{Z}$, both* $\mathrm{Tor}_n^R(M,\mathrm{K}[\mathbf{x};N])$ *and* $\mathrm{Ext}_R^n(M,\mathrm{K}[\mathbf{x};N])$ *are finite S-modules annihilated by* $\mathbf{x}S + \mathrm{Ann}_{\mathcal{D}(R)}(M)S + \mathrm{Ann}_{\mathcal{D}(S)}(N)$.

Proof. The complex $\mathrm{K}[\mathbf{x};N]$) is homologically finite by (1.5.5), so the assertion about finiteness follows from Lemma (1.3.2). The assertion about annihilation comes from (1.5.3) and (1.4.2). □

2. Dimensions. We extend some well known theorems on modules of finite injective dimension, namely, the Bass Equality, see [26, (18.9)], and the "Bass

Conjecture" proved by P. Roberts, see [30, §3.1] and [31, (13.4)]. The novelty is that finiteness over R is relaxed to finiteness over S. Khatami and Yassemi [21, (3.5)], and Takahashi and Yoshino [35, (5.2)] have independently obtained the first equality below.

THEOREM 2.1. *If L is a finite S-module and* $\operatorname{inj\,dim}_R L < \infty$*, then*

$$\operatorname{inj\,dim}_R L = \operatorname{depth} R = \dim R.$$

This is proved at the end of the section, after we engineer a situation where the original results apply. We recall how some classical concepts extend to complexes.

Let M be a homologically bounded complex of R-modules. Its *flat dimension* and its *injective dimension* over R are, respectively, the numbers

$$\operatorname{flat\,dim}_R M = \sup\left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{l} Y_n \neq 0 \text{ for some bounded complex} \\ \text{of flat } R\text{-modules } Y \text{ with } Y \simeq M \end{array} \right\} \quad \text{and}$$

$$\operatorname{inj\,dim}_R M = -\inf\left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{l} Y_n \neq 0 \text{ for some bounded complex} \\ \text{of injective } R\text{-modules } Y \text{ with } Y \simeq M \end{array} \right\}.$$

The nth *Betti number* $\beta_n^R(M)$ and the nth *Bass number* $\mu_R^n(M)$ are defined to be

$$\beta_n^R(M) = \operatorname{rank}_k \operatorname{Tor}_n^R(k, M) \qquad \text{and} \qquad \mu_R^n(M) = \operatorname{rank}_k \operatorname{Ext}_R^n(k, M).$$

2.2. By [8, (5.5)], for the homologically finite complex N, one has

$$\operatorname{flat\,dim}_R N = \sup\{n \in \mathbb{Z} \mid \beta_n^R(N) \neq 0\}$$
$$\operatorname{inj\,dim}_R N = \sup\{n \in \mathbb{Z} \mid \mu_R^n(N) \neq 0\}.$$

Next we present a construction of Avramov, Foxby, and B. Herzog [10], which is used in proofs of many theorems throughout the paper.

2.3. Cohen factorizations. Let $\grave{\varphi}\colon R \to \widehat{S}$ denote the composition of φ with the completion map $S \to \widehat{S}$. A *Cohen factorization* of $\grave{\varphi}$ is a commutative diagram

$$\begin{tikzcd} & R' \arrow[dr, "\varphi'"] & \\ R \arrow[ur, "\dot{\varphi}"] \arrow[rr, "\grave{\varphi}"] & & \widehat{S} \end{tikzcd}$$

of local homomorphisms such that the map $\dot{\varphi}$ is flat, the ring R' is complete, the ring $R'/\mathfrak{m}R'$ is regular, and the map φ' is surjective.

Clearly, the homomorphisms and rings in the diagram above satisfy

$$\operatorname{edim}\varphi \leq \operatorname{edim}\dot{\varphi} = \dim(R'/\mathfrak{m}R')$$

When equality holds the factorization is said to be *minimal*. It is proved in [10, (1.5)] that the homomorphism $\grave{\varphi}$ always has a minimal Cohen factorization.

For the rest of this subsection we fix a minimal Cohen factorization of $\grave{\varphi}$. With the next result we lay the groundwork for several proofs in this paper.

PROPOSITION 2.4. *For each minimal set* $\mathbf{x}$ *of generators of* $\mathfrak{n}$ *modulo* $\mathfrak{m}S$ *there are isomorphisms in the derived category of S-modules, as follows:*

$$k \otimes^{\mathbf{L}}_R (\mathrm{K}[\mathbf{x}; N]) \simeq \mathrm{K}[\mathbf{x}; k \otimes^{\mathbf{L}}_R N] \simeq l \otimes^{\mathbf{L}}_{R'} \widehat{N}$$
$$\mathbf{R}\mathrm{Hom}_R(k, \mathrm{K}[\mathbf{x}; N]) \simeq \mathrm{K}[\mathbf{x}; \mathbf{R}\mathrm{Hom}_R(k, N)] \simeq \Sigma^{\operatorname{edim}\varphi}\, \mathbf{R}\mathrm{Hom}_{R'}(l, \widehat{N}).$$

Proof. We give the proof for homomorphisms and omit the other, similar, argument.

Considering S as a subset of $\widehat{S}$, lift $\mathbf{x}$ to a subset $\mathbf{x}'$ of R', containing $\operatorname{edim}\varphi$ elements. As the ring $P = R'/\mathfrak{m}R'$ is regular, the image of $\mathbf{x}'$ in P is a regular system of parameters. Thus, the Koszul complex $\mathrm{K}[\mathbf{x}'; P]$ is a free resolution of l, that is, $\mathrm{K}[\mathbf{x}'; P] \simeq l$. This accounts for the first isomorphism in the chain

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_{R'}(l, \widehat{N}) &\simeq \mathbf{R}\mathrm{Hom}_{R'}\left(\mathrm{K}[\mathbf{x}'; P], \widehat{N}\right)\\
&\simeq \mathbf{R}\mathrm{Hom}_{R'}\left(P, \mathbf{R}\mathrm{Hom}_{R'}(\mathrm{K}[\mathbf{x}'; R'], \widehat{N})\right)\\
&\simeq \mathbf{R}\mathrm{Hom}_{R'}\left(R' \otimes_R k, \mathrm{Hom}_{R'}(\mathrm{K}[\mathbf{x}'; R'], \widehat{N})\right)\\
&\simeq \mathbf{R}\mathrm{Hom}_{R}\left(k, \mathrm{Hom}_{R'}(\mathrm{K}[\mathbf{x}'; R'], \widehat{N})\right)\\
&\simeq \mathbf{R}\mathrm{Hom}_{R}\left(k, \mathrm{Hom}_{\widehat{S}}(\mathrm{K}[\mathbf{x}; \widehat{S}], \widehat{N})\right)\\
&\simeq \mathbf{R}\mathrm{Hom}_{R}\left(k, \mathrm{Hom}_{S}(\mathrm{K}[\mathbf{x}; S], \widehat{N})\right).
\end{aligned}$$

The third isomorphism holds because $\mathrm{K}[\mathbf{x}'; R']$ is a finite complex of free R'-modules, while the remaining ones are adjunctions.

The first isomorphism below holds because $\mathrm{K}[\mathbf{x}; S]$ is a finite free complex:

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R\left(k, \mathrm{Hom}_S(\mathrm{K}[\mathbf{x}; S], \widehat{N})\right) &\simeq \mathbf{R}\mathrm{Hom}_R\left(k, \mathrm{Hom}_S(\mathrm{K}[\mathbf{x}; S], N) \otimes_S \widehat{S}\right)\\
&\simeq \mathbf{R}\mathrm{Hom}_R(k, \mathrm{Hom}_S(\mathrm{K}[\mathbf{x}; S], N)) \otimes_S \widehat{S}.
\end{aligned}$$

The second one does because $\widehat{S}$ is a flat S-module, $\operatorname{Hom}_S(\mathrm{K}[\mathbf{x};S],N)$ has bounded homology, and k has a resolution by finite free R-modules.

As $\widehat{S}$ is flat over S, and each $\mathrm{H}_n\mathbf{R}\operatorname{Hom}_R(k,\operatorname{Hom}_S(\mathrm{K}[\mathbf{x};S],N))$ has finite length over S by (1.5.6), the canonical morphism

$$\mathbf{R}\operatorname{Hom}_R(k,\operatorname{Hom}_S(\mathrm{K}[\mathbf{x};S],N))\otimes_S\widehat{S}\longleftarrow\mathbf{R}\operatorname{Hom}_R(k,\operatorname{Hom}_S(\mathrm{K}[\mathbf{x};S],N))$$

is an isomorphism in $\mathcal{D}(S)$. Furthermore, we have isomorphisms

$$\begin{aligned}\mathbf{R}\operatorname{Hom}_R(k,\operatorname{Hom}_S(\mathrm{K}[\mathbf{x};S],N)) &\simeq \mathbf{R}\operatorname{Hom}_R(k,\Sigma^{-\operatorname{edim}\varphi}\mathrm{K}[\mathbf{x};N])\\ &\simeq \Sigma^{-\operatorname{edim}\varphi}\mathbf{R}\operatorname{Hom}_R(k,\mathrm{K}[\mathbf{x};N])\\ &\simeq \Sigma^{-\operatorname{edim}\varphi}\mathrm{K}[\mathbf{x};\mathbf{R}\operatorname{Hom}_R(k,N)]\end{aligned}$$

obtained from (1.5.2), the definition of Σ, and (1.5.1), respectively. Linking the chains above we obtain the desired isomorphisms. □

Our first application of the proposition above is to homological dimensions. The assertions on the flat dimension of $\widehat{N}$ over R' are known, see [10, (3.2)].

Corollary 2.5. *The following (in)equalities hold:*

$$\operatorname{flat\,dim}_R N\le\operatorname{flat\,dim}_{R'}\widehat{N}\le\operatorname{flat\,dim}_R N+\operatorname{edim}\varphi$$
$$\operatorname{inj\,dim}_{R'}\widehat{N}=\operatorname{inj\,dim}_R N+\operatorname{edim}\varphi.$$

Proof. Let $\mathbf{x}$ be a minimal generating set for $\mathfrak{n}$ modulo $\mathfrak{m}S$. From the proposition and the expressions for homological dimensions in (2.2), one gets

$$\operatorname{flat\,dim}_{R'}\widehat{N}=\sup\mathrm{H}\left(\mathrm{K}[\mathbf{x};k\otimes^{\mathbf{L}}_R N]\right)$$
$$\operatorname{inj\,dim}_{R'}\widehat{N}=-\inf\mathrm{H}(\mathrm{K}[\mathbf{x};\mathbf{R}\operatorname{Hom}_R(k,N)])+\operatorname{edim}\varphi.$$

As $\operatorname{edim}\varphi=\operatorname{card}(\mathbf{x})$, and the S-modules $\mathrm{H}_n(k\otimes^{\mathbf{L}}_R N)$ and $\mathbf{R}\operatorname{Hom}_R(k,N)$ are finite for each n by (1.3.2), so (1.2.2) gives the desired (in)equalities. □

The interval for the flat dimension of $\widehat{N}$ over R' cannot be narrowed, in general, as the following special case demonstrates.

Example 2.6. If R is a field, the ring S is complete and regular, and N is an S-module, then the statement above reduces to inequalities $0\le\operatorname{flat\,dim}_S N\le\dim S$.

Proof of Theorem 2.1. Let $R\to R'\to\widehat{S}$ be a minimal Cohen factorization of $\grave{\varphi}$. Corollary 2.5 and the minimality of the factorization yield

$$\begin{aligned}\operatorname{inj\,dim}_R L &= \operatorname{inj\,dim}_{R'}\widehat{L}-\operatorname{edim}\varphi\\ &= \operatorname{inj\,dim}_{R'}\widehat{L}-\operatorname{edim}(R'/\mathfrak{m}R').\end{aligned}$$

In particular, inj $\dim_{R'} \widehat{L}$ is finite, so Bass' Formula gives the first equality below. The other two are due to the regularity of $R'/\mathfrak{m}R'$, and the flatness of R' over R:

$$\begin{aligned}
\operatorname{inj}\dim_{R'} \widehat{L} - \operatorname{edim}(R'/\mathfrak{m}R') &= \operatorname{depth} R' - \operatorname{edim}(R'/\mathfrak{m}R') \\
&= \operatorname{depth} R' - \operatorname{depth}(R'/\mathfrak{m}R') \\
&= \operatorname{depth} R.
\end{aligned}$$

Roberts' Theorem shows that R' is Cohen-Macaulay; by flat descent, so is R. □

3. Koszul invariants. Koszul complexes on minimal sets of generators of the maximal ideal $\mathfrak{n}$ of S appear systematically in our study. This section is devoted to establishing relevant properties of such complexes. While several of them are known, in one form or another, we have not been able to find in the literature statements presenting the detail and generality needed below. It should be noted that the invariants discussed in this section depend only on the action of S on N, while R plays no role.

LEMMA 3.1. *Set $s = \operatorname{edim} S$, let $\mathbf{w}$ be a minimal set of generators of $\mathfrak{n}$, and let $\mathbf{z}$ be a set of generators of $\mathfrak{n}$ of cardinality u.*

Let A be the Koszul complex of a set consisting of $(u-s)$ zeros.

(1) *There is an isomorphism of DG algebras $\mathrm{K}[\mathbf{z};S] \cong \mathrm{K}[\mathbf{w};S] \otimes_S A$*

(2) *If $\mathbf{z}$ minimally generates $\mathfrak{n}$, then $\mathrm{K}[\mathbf{z};S] \cong \mathrm{K}[\mathbf{w};S]$.*

(3) *The complexes of S-modules $\mathrm{K}[\mathbf{z};N]$ and $\mathrm{K}[\mathbf{w};N] \otimes_S A$ are isomorphic.*

Proof. (1) Since $\mathbf{z} \subseteq (\mathbf{w})$ we may write

$$z_j = \sum_{i=1}^{s} a_{ij} w_i \quad \text{with} \quad a_{ij} \in S \quad \text{for} \quad j = 1, \dots, u.$$

Renumbering the elements of $\mathbf{z}$ if necessary, we may assume that the matrix $(a_{ij})_{1\leqslant i,j\leqslant s}$ is invertible. Choose bases $\{e_1, \dots, e_s\}$ and $\{f_1, \dots, f_u\}$ of the degree 1 components of $\mathrm{K}[\mathbf{w};S]$ and $\mathrm{K}[\mathbf{z};S]$, respectively, such that

$$\partial(e_i) = w_i \text{ for } 1 \le i \le s \quad \text{and} \quad \partial(f_j) = z_j \text{ for } 1 \le j \le u.$$

Let $\{e_{s+1}, \dots, e_u\}$ be a basis for the free module A_1. The graded algebra underlying $\mathrm{K}[\mathbf{z};S]$ is the exterior algebra on $\{f_1, \dots, f_u\}$, so there exists a unique morphism $\varkappa\colon \mathrm{K}[\mathbf{z};S] \to \mathrm{K}[\mathbf{w};S] \otimes_S A$ of graded algebras over S, such that

$$\varkappa(f_j) = \begin{cases} \sum_{i=1}^{s} a_{ij}(e_i \otimes 1) & \text{for} \quad 1 \le j \le s \\ \sum_{i=1}^{s} a_{ij}(e_i \otimes 1) - (1 \otimes e_j) & \text{for} \quad s < j \le u. \end{cases}$$

Note that $\varkappa$ is an isomorphism in degree 1. The graded algebra underlying $\mathrm{K}[\mathbf{w}; S]\otimes_S A$ is the exterior algebra on $\{e_1\otimes 1, \dots, e_s\otimes 1, 1\otimes e_{s+1}, \dots, 1\otimes e_u\}$, so $\varkappa$ is an isomorphism of graded algebras. The formulas above yield $\partial\varkappa(f_j) = \varkappa\partial(f_j)$ for $j = 1, \dots, u$. It follows that $\varkappa$ is a morphism of DG algebras.

The assertions in (2) and (3) are consequences of (1). □

3.2. We let $\mathrm{K}^S[N]$ denote the complex $\mathrm{K}[\mathbf{w}; N]$ on a minimal generating set of $\mathfrak{n}$. There is little ambiguity in the notation because different choices of $\mathbf{w}$ yield isomorphic complexes; see Lemma 3.1.2. We set $K^S = \mathrm{K}^S[S]$.

Lemma 1.5.6 shows that $\mathrm{H}_n(\mathrm{K}^S[N])$ is a finite l-vector space for each $n \in \mathbb{Z}$, which is trivial for all $|n| \gg 0$. Thus, one can form a Laurent polynomial

$$K_N^S(t) = \sum_{n\in\mathbb{Z}} \kappa_n^S(N)\, t^n \qquad \text{where} \qquad \kappa_n^S(N) = \operatorname{rank}_l \mathrm{H}_n(\mathrm{K}^S[N])$$

with nonnegative coefficients. We call it the *Koszul polynomial* of N over S.

Some numbers canonically attached to N play a role in further considerations.

3.3. The *depth* and the *type* of N over S are defined, respectively, to be the numbers

$$\operatorname{depth}_S N = \inf\{n \in \mathbb{Z} \mid \operatorname{Ext}_S^n(l, N) \neq 0\}$$
$$\operatorname{type}_S N = \operatorname{rank}_l \operatorname{Ext}_S^{\operatorname{depth}_S N}(l, N).$$

When N is an S-module the expressions above yield the familiar invariants. By [19, (6.5)], depth can be computed from a Koszul complex, namely

$$\operatorname{depth}_S N = \operatorname{edim} S - \sup \mathrm{H}\left(\mathrm{K}^S[N]\right).$$

3.4. A *Cohen presentation* of $\widehat{S}$ is an isomorphism $\widehat{S} \cong T/\mathfrak{b}$ where $(T, \mathfrak{r}, l)$ is a regular local ring. Cohen's Structure Theorem proves one always exists. It can be taken to be *minimal*, in the sense that $\operatorname{edim} T = \operatorname{edim} S$: If not, pick $x \in \mathfrak{b} \setminus \mathfrak{r}^2$, note the isomorphism $\widehat{S} \cong (T/xT)/(\mathfrak{b}/xT)$ where T/xT is regular, and iterate.

In the next result, and on several occasions in the sequel, we use the theory of dualizing complexes, for which we refer to Hartshorne [18, V.2].

3.5. Let Q be a local ring with residue field h. A *normalized dualizing complex* D for Q is a homologically finite complex of Q-modules E, such that

$$\mathbf{R}\mathrm{Hom}_Q(h, E) \simeq h$$

These properties describe E uniquely up to isomorphism in $\mathcal{D}(Q)$. When E is a normalized dualizing complex, for every complex of Q-modules X we set

$$X^\dagger = \mathbf{R}\mathrm{Hom}_Q(X, E)$$

If X is homologically finite, then so is $X^\dagger$, and the canonical morphism $X \to X^{\dagger\dagger}$ is an isomorphism in $\mathcal{D}(Q)$. When necessary, we write $X^\dagger_Q$ instead of $X^\dagger$.

A local ring P is Gorenstein if and only if $\Sigma^{\dim P} P$ is a normalized dualizing complex for P. If this is the case and $P \to Q$ is a surjective homomorphism, then $\mathbf{R}\mathrm{Hom}_P\left(Q, \Sigma^{\dim P} P\right)$ is a normalized dualizing complex for Q. Thus, Cohen's Structure Theorem implies that every complete local ring has a dualizing complex.

3.6. The complexes of S-modules $\mathrm{K}^S[N^\dagger]^\dagger$ and $\Sigma^{-\operatorname{edim} S}\mathrm{K}^S[N]$ are isomorphic in $\mathcal{D}(R)$.

Indeed, if D is a normalized dualizing complex for S, then one has

$$\begin{aligned}
\mathrm{Hom}_S(\mathrm{K}^S[N^\dagger], D) &\simeq \mathrm{Hom}_S(K^S \otimes_S N^\dagger, D)\\
&\simeq \mathrm{Hom}_S(K^S, N^{\dagger\dagger})\\
&\simeq \mathrm{Hom}_S(K^S, N)\\
&\simeq \Sigma^{-\operatorname{edim} S}\mathrm{K}^S[N].
\end{aligned}$$

We set $\operatorname{codim} S = \operatorname{edim} S - \dim S$. If L is an S-module, then $\nu_S L$ denotes the minimal number of generators of L. The following result collects most of the arithmetic information required on the Koszul polynomial.

THEOREM 3.7. *Assume* $\mathrm{H}(N) \neq 0$*, and set*

$$s = \operatorname{edim} S\,, \quad c = \operatorname{codim} S\,, \quad i = \inf \mathrm{H}(N)\,, \quad m = \nu_S \mathrm{H}_i(N)\,, \quad g = \operatorname{depth}_S N\,.$$

The Koszul polynomial $K^S_N(t)$ *has the following properties.*

(1) $\operatorname{ord} K^S_N(t) = i$ *and* $\kappa^S_i(N) = m$.

(2) $\deg K^S_N(t) = (s - g)$ *and* $\kappa^S_{s-g}(N) = \operatorname{type}_S N$.

(3) $K^S_N(t) = K^{\widehat{S}}_{\widehat{N}}(t) = P^T_{\widehat{N}}(t)$, *where* $\widehat{S} \cong T/\mathfrak{b}$ *is a minimal Cohen presentation.*

(4) $K^{\widehat{S}}_{\widehat{N}^\dagger}(t) = t^s K^S_N(t^{-1})$.

If S is not regular, then the following also hold.

(5) $\kappa^S_{i+1}(N) \geq \kappa^S_i(N) + c - 1$.

(6) $\kappa^S_{s-g-1}(N) \geq \kappa^S_{s-g}(N) + c - 1$.

(7) $K^S_N(t) = (1 + t) \cdot L(t)$ *where* $L(t)$ *is a Laurent polynomial of the form*

$$L(t) = mt^i + \text{higher order terms with nonnegative coefficients.}$$

Proof. (1) The equality $\operatorname{ord} K^S_N(t) = i$ comes from (1.2.2).
The equality $\kappa^S_i(N) = m$ results from the canonical isomorphisms

$$\mathrm{H}_i(\mathrm{K}^S[N]) \cong \mathrm{H}_i(K^S \otimes_S N) \cong \mathrm{H}_0(K^S) \otimes_S \mathrm{H}_i(N) = l \otimes_S \mathrm{H}_i(N).$$

(2) The equality $\deg K^S_N(t) = (s-g)$ comes from (3.3).
Because $\mathfrak{n}$ annihilates each $\mathrm{H}_q(K^S)$, the standard spectral sequence with

$$E^2_{pq} = \operatorname{Ext}^{-p}_S(\mathrm{H}_q(K^S), N) \Rightarrow \mathrm{H}_{p+q} \operatorname{Hom}_S(K^S, N)$$

has $E^2_{pq} = 0$ for $p > -g$. It has a corner, which yields the first isomorphism below

$$\operatorname{Ext}^g_S(l, N) = E^2_{-g\,0} \cong \mathrm{H}_{-g} \operatorname{Hom}_S(K^S, N) \cong \mathrm{H}_{s-g}(\mathrm{K}^S[N]).$$

The second one is from (1.5.2). Thus, we get $\kappa^S_{s-g}(N) = \operatorname{type}_S N$.

(3) Any minimal generating set $\mathbf{w}$ of $\mathfrak{n}$ minimally generates the maximal ideal of $\widehat{S}$, and the canonical map $\mathrm{K}[\mathbf{w}; N] \to \mathrm{K}[\mathbf{w}; \widehat{N}]$ is an isomorphism in $\mathcal{D}(S)$.

Having proved the first equality, for the rest of the proof we may assume that S is complete. We choose an isomorphism $S \cong T/\mathfrak{b}$, where $(T, \mathfrak{r}, l)$ is a regular local ring with $\dim T = \operatorname{edim} S$, and a normalized dualizing D complex for S.

For the second equality in (3), note that a minimal generating set of $\mathfrak{r}$ maps to a minimal generating set of $\mathfrak{n}$. As T is regular, K^T is a free resolution of l over T. Now invoke the isomorphism $K^T \otimes_T N \cong \mathrm{K}^S[N]$.

(4) There is a spectral sequence with

$$E^2_{pq} = \operatorname{Ext}^{-p}_S(\mathrm{H}_q(\mathrm{K}^S[N^\dagger]), D) \Rightarrow \mathrm{H}_{p+q} \operatorname{Hom}_S(\mathrm{K}^S[N^\dagger], D).$$

As $\mathrm{H}(\mathrm{K}^S[N^\dagger])$ is a finite dimensional graded l-vector space, the defining property of D implies $E^2_{pq} = 0$ for $p \neq 0$, and yields for all n isomorphisms

$$\operatorname{Hom}_l(\mathrm{H}_n(\mathrm{K}^S[N^\dagger]), l) \cong \mathrm{H}_n \operatorname{Hom}_S(\mathrm{K}^S[N^\dagger], D)$$

of l-vector spaces. On the other hand, (3.6) yields for all n isomorphisms

$$\mathrm{H}_n \operatorname{Hom}_S(\mathrm{K}^S[N^\dagger], D) \cong \mathrm{H}_{s-n}(\mathrm{K}^S[N])$$

of l-vector spaces. Comparison of ranks produces the desired equalities.

As N is homologically finite, it is isomorphic in $\mathcal{D}(T)$ to a finite free complex of T-modules G that satisfies $\partial(G) \subseteq \mathfrak{r}G$; see [30, (II.2.4)]. From (3) we get

$$\operatorname{rank}_T(G_n) = \kappa^S_n(N) \qquad \text{for all} \qquad n \in \mathbb{Z}.$$

We use the complex G in the proofs of the remaining statements.

(5) As $\mathrm{H}(G) \cong \mathrm{H}(N)$ and $\inf \mathrm{H}(N) = i$, there is an exact sequence of T-modules:

$$G_{i+1} \longrightarrow G_i \longrightarrow \mathrm{H}_i(N) \longrightarrow 0.$$

The Generalized Principal Ideal Theorem gives the first inequality below:

$$\begin{aligned} \kappa^S_{i+1}(N) - \kappa^S_i(N) + 1 &\geq \operatorname{height} \operatorname{Ann}_T(\mathrm{H}_i(N)) \\ &= \dim T - \dim_T \mathrm{H}_i(N) \\ &\geq \dim T - \dim S \\ &= \operatorname{edim} S - \dim S, \end{aligned}$$

see [26, (13.10)]. The first equality holds because T is regular, and hence catenary and equidimensional, while the other relations are clear.

(6) In view of (4), the inequality (5) for the complex $N^\dagger$ yields the desired inequality, $\kappa^S_{s-g-1}(N) \geq \kappa^S_{s-g}(N) + c - 1$.

(7) Let U be the field of fractions of T. As S is not regular, $\mathfrak{b}$ is not zero, so $N \otimes_T U = 0$. Also, $G \otimes_T U \simeq N \otimes_T U$ holds in $\mathcal{D}(U)$, so $G \otimes_T U$ is an exact complex of finite U-vector spaces, and hence is the mapping cone of the identity map of a complex W with trivial differential. For $L(t) = \sum_{n\in\mathbb{Z}} \operatorname{rank}_U(W_n)\, t^n$ one gets

$$\begin{aligned} K^S_N(t) = P^T_N(t) &= \sum_{n\in\mathbb{Z}} \operatorname{rank}_T(G_n)\, t^n \\ &= \sum_{n\in\mathbb{Z}} \operatorname{rank}_U(G \otimes_T U)_n\, t^n \\ &= \sum_{n\in\mathbb{Z}} \operatorname{rank}_U(W_n)\, t^n \cdot (1+t). \end{aligned}$$

It follows that $L(t)$ has order i and starts with mt^i. □

An important invariant of Koszul complexes is implicit in a result of Serre.

3.8. Let L be a finite S-module. Set $K = \mathrm{K}[\mathbf{w}; L]$, where $\mathbf{w} = \{w_1, \dots, w_s\}$ minimally generates $\mathfrak{n}$, and $\mathfrak{n}^j = S$ for $j \leq 0$. As $\partial(\mathfrak{n}^j K_n) \subseteq \mathfrak{n}^{j+1} K_{n-1}$ for all j and n, for each $i \in \mathbb{Z}$ one has a complex of S-modules

$$I^i_L = \quad 0 \longrightarrow \mathfrak{n}^{i-s} K_s \longrightarrow \mathfrak{n}^{i-s+1} K_{s-1} \longrightarrow \cdots \longrightarrow \mathfrak{n}^{i-1} K_1 \longrightarrow \mathfrak{n}^i K_0 \to 0.$$

Lemma 3.1.2 shows that it does not depend on the choice of $\mathbf{w}$, up to isomorphism. We define the *spread* of N over S to be the number

$$\operatorname{spread}_S L = \inf\{i \in \mathbb{Z} \mid \mathrm{H}(I_L^j) = 0 \text{ for all } j \geq i\}.$$

Serre [33, (IV.A.3)] proves that it is finite. We write $\operatorname{spread} S$ in place of $\operatorname{spread}_S S$.

In some cases the new invariant is easy to determine.

Example 3.9. If $\mathfrak{n}^v L = 0 \neq \mathfrak{n}^{v-1} L$, then $\operatorname{spread}_S L = \operatorname{edim} S + v$.
Indeed, for $s = \operatorname{edim} S$ one has $I_L^{s+v} = 0$ and $\mathrm{H}_s(I_L^{s+v-1}) \supseteq \mathfrak{n}^{v-1} K_s \neq 0$.

To deal with more involved situations we interpret $\operatorname{spread}_S N$ in terms of associated graded modules and their Koszul complexes.

3.10. Given an S-module L we set $\mathfrak{n}^i L = L$ for all $i \leq 0$ and write $\operatorname{gr}_{\mathfrak{n}}(L)$ for the graded abelian group associated to the $\mathfrak{n}$-adic filtration $\{\mathfrak{n}^i L\}_{i\in\mathbb{Z}}$; it is a graded module over the graded ring $\operatorname{gr}_{\mathfrak{n}}(S)$. Note that $\mathbf{w}^* = \{w_i + \mathfrak{n}^2, \dots, w_s + \mathfrak{n}^2\} \subseteq \operatorname{gr}_{\mathfrak{n}}(S)_1$ minimally generates $\operatorname{gr}_{\mathfrak{n}}(S)$ as an algebra over $\operatorname{gr}_{\mathfrak{n}}(S)_0 = l$.

The Koszul complex $\mathrm{K}[\mathbf{w}^*; \operatorname{gr}_{\mathfrak{n}}(S)]$ becomes a complex of graded $\operatorname{gr}_{\mathfrak{n}}(S)$-modules by assigning bidegree $(1, 1)$ to each generator of $\mathrm{K}[\mathbf{w}^*; \operatorname{gr}_{\mathfrak{n}}(S)]_1$. If $\mathbf{w}'$ also is a minimal generating set for $\mathfrak{n}$, then the complexes of S-modules $\mathrm{K}[\mathbf{w}; S]$ and $\mathrm{K}[\mathbf{w}'; S]$ are isomorphic, see Lemma 3.1.2; any such isomorphism induces an isomorphism $\mathrm{K}[\mathbf{w}'^*; \operatorname{gr}_{\mathfrak{n}}(S)] \cong \mathrm{K}[\mathbf{w}^*; \operatorname{gr}_{\mathfrak{n}}(S)]$ of complexes of graded $\operatorname{gr}_{\mathfrak{n}}(S)$-modules. We let $K^{\operatorname{gr}_{\mathfrak{n}}(S)}$ denote any such complex, and form the complex of graded $\operatorname{gr}_{\mathfrak{n}}(S)$-modules

$$\mathrm{K}^{\operatorname{gr}_{\mathfrak{n}}(S)}[\operatorname{gr}_{\mathfrak{n}}(L)] = K^{\operatorname{gr}_{\mathfrak{n}}(S)} \otimes_{\operatorname{gr}_{\mathfrak{n}}(S)} \operatorname{gr}_{\mathfrak{n}}(L).$$

In the next proposition we refine Serre's result (3.8). Rather than using the result itself, we reinterpret some ideas of its proof in our argument.

Proposition 3.11. *The following equality holds:*

$$\operatorname{spread}_S L = \sup\{i \in \mathbb{Z} \mid \mathrm{H}_n\left(\mathrm{K}^{\operatorname{gr}_{\mathfrak{n}}(S)}[\operatorname{gr}_{\mathfrak{n}}(L)]\right)_i \neq 0 \textit{ for some } n \in \mathbb{Z}\}.$$

Proof. Set $I = I_L$. By construction, we have isomorphisms

$$\bigoplus_{i\in\mathbb{Z}} I^i/I^{i+1} \cong \mathrm{K}[\mathbf{w}^*; \operatorname{gr}_{\mathfrak{n}}(L)]$$

of complexes of graded ℓ-vector spaces. Thus, to prove the proposition it suffices to show that the following claim holds for each integer a.

Claim. $\mathrm{H}(I^i) = 0$ for $i \geq a$ if and only if $\mathrm{H}(I^i/I^{i+1}) = 0$ for $i \geq a$.

Indeed, the exact sequence of complexes $0 \to I^{i+1} \to I^i \to I^i/I^{i+1} \to 0$ shows that if $\mathrm{H}(I^i)$ vanishes for $i \geq a$, then $\mathrm{H}(I^i/I^{i+1}) = 0$ for $i \geq a$. Conversely, assume that the latter condition holds. Induction on j using the exact sequences

$$0 \to I^{a+j}/I^{a+j+1} \to I^a/I^{a+j+1} \to I^a/I^{a+j} \to 0$$

of complexes yields $\mathrm{H}(I^a/I^{a+j}) = 0$. Since the inverse system of complexes

$$\cdots \to I^a/I^{a+j+1} \to I^a/I^{a+j} \to \cdots \to I^a/I^{a+2} \to I^a/I^{a+1} \to 0$$

is surjective, one gets $\mathrm{H}(\varprojlim_j I^a/I^{a+j}) = 0$. Now $(I^{a+j})_n = \mathfrak{n}^{a+j-n}K_n$ for each n, so $\varprojlim_j (I^a/I^{a+j})_n$ is the $\mathfrak{n}$-adic completion of the S-module $\mathfrak{n}^{a-n}K_n$. This module is finite, so its completion is isomorphic to $(\mathfrak{n}^{a-n}K_n) \otimes_S \widehat{S}$. The upshot is that

$$\varprojlim_j \left(I^a/I^{a+j} \right) \cong I^a \otimes_S \widehat{S}$$

as complexes of S-modules. From here and the flatness of $\widehat{S}$ over S one gets

$$\mathrm{H}(I^a) \otimes_S \widehat{S} \cong \mathrm{H}(I^a \otimes_S \widehat{S}) = \mathrm{H}\left(\varprojlim_j I^a/I^{a+j} \right) = 0.$$

Since the S-module $\widehat{S}$ is also faithful, we deduce $\mathrm{H}(I^a) = 0$, as claimed. □

There is another interpretation of the numbers in the preceding proposition.

Remark 3.12. Let $B = \{B_j\}_{i\geqslant 0}$ be a graded ring with $B_0 \cong l$, finitely generated as a B_0-algebra, and let M be a graded and finite B-module. The number

$$a_B^*(M) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}^n_{\mathfrak{b}}(M)_i \neq 0 \text{ for some } n \in \mathbb{Z}\}$$

where $\mathfrak{b}$ is the irrelevant maximal ideal $B_{\geqslant 1}$ of B, and $\mathrm{H}^n_{\mathfrak{b}}(M)$ is the nth local cohomology module of M, has been studied by Trung [36] and others. We claim: if b is the minimal number of homogeneous generators of the l-algebra B, then

$$\sup\{i \in \mathbb{Z} \mid \mathrm{H}_n(\mathrm{K}^B[M])_i \neq 0 \text{ for some } n \in \mathbb{Z}\} = a_B^*(M) + b.$$

This preceding identity can be verified by using a spectral sequence with

$$E_2^{pq} = \mathrm{H}_p\left(\mathrm{K}^B[\mathrm{H}_{\mathfrak{b}}^{-q}(M)] \right) \Rightarrow \mathrm{H}_{p+q}(\mathrm{K}^B[M])$$

that lies in the fourth quadrant and has differentials $d_r^{pq}\colon E_r^{pq} \to E_r^{p-r,q+r-1}$.

4. Sequences. The flat dimension and the injective dimension of N over R can be determined from the *number* of non-vanishing groups $\operatorname{Tor}_n^R(k,N)$ and $\operatorname{Ext}_R^n(k,N)$, as recalled in (2.2). When this number is infinite, we propose to use the *sizes* of these groups as a measure of the homological intricacy of the complex N over R. Each such group carries a canonical structure of *finite S-module*, so its size is reflected in a number of natural invariants, such as minimal number of generators, multiplicity, rank, or length. Among these it is the length over S, denoted ℓ_S, that has the best formal properties, but it is of little use unless some extra hypothesis (for instance, that the ring $S/\mathfrak{m}S$ is artinian) guarantees its finiteness.

To overcome this problem, we take a cue from Serre's approach to multiplicities and replace N by an appropriate Koszul complex. As there is no canonical choice of such a complex, the question arises whether the new invariants are well defined. In this section we prove that indeed they are, and describe them in alternative terms.

4.1. Betti numbers and Bass numbers. Let $\mathbf{x}$ be a minimal generating set for $\mathfrak{n}$ modulo $\mathfrak{m}S$. Lemma 1.5.6 shows that $\operatorname{Tor}_n^R(k, \mathrm{K}[\mathbf{x};N])$ and $\operatorname{Ext}_R^n(k, \mathrm{K}[\mathbf{x};N])$ are finite S-modules annihilated by $\mathfrak{n}$, hence they are l-vector spaces of finite rank.

For each $n \in \mathbb{Z}$ we define the nth *Betti number* $\beta_n^{\varphi}(N)$ *of N over* φ and the nth *Bass number* $\mu_{\varphi}^n(N)$ *of N over* φ by the formulas

$$\begin{aligned}\beta_n^{\varphi}(N) &= \operatorname{rank}_l \operatorname{Tor}_n^R(k, \mathrm{K}[\mathbf{x};N])\\ \mu_{\varphi}^n(N) &= \operatorname{rank}_l \operatorname{Ext}_R^{n-\operatorname{edim}\varphi}(k, \mathrm{K}[\mathbf{x};N]).\end{aligned}$$

These numbers are invariants of N: pick a minimal Cohen factorization $R \to R' \to \widehat{S}$ of $\acute{\varphi}$ and apply Proposition 2.4 to get isomorphisms of S-modules

$$\begin{aligned}\operatorname{Tor}_n^R(k, \mathrm{K}[\mathbf{x};N]) &\cong \operatorname{Tor}_n^{R'}(\ell, \widehat{N})\\ \operatorname{Ext}_R^{n-\operatorname{edim}\varphi}(k, \mathrm{K}[\mathbf{x};N]) &\cong \operatorname{Ext}_{R'}^n(\ell, \widehat{N}).\end{aligned} \tag{4.1.0}$$

The vector spaces on the right do not depend on $\mathbf{x}$, so neither do their ranks. Such an independence may not be taken for granted, because $\mathrm{K}[\mathbf{x};N]$ is not defined uniquely up to isomorphism in $\mathcal{D}(S)$; even its homology may depend on $\mathbf{x}$.

Example 4.1.1. Let k be a field, set $R = k[[u]]$ and $S = k[[x,y]]/(xy)$, and let $\varphi\colon R \to S$ be the homomorphism of complete k-algebras defined by $\varphi(u) = y$. The Koszul complexes on $\mathbf{x} = \{x\}$ and $\mathbf{x}' = \{x+y\}$ satisfy

$$\mathrm{H}_n(\mathrm{K}[\mathbf{x};S]) \cong \begin{cases} S/(x) & \text{for } n = 0,1\\ 0 & \text{otherwise}\end{cases} \qquad \mathrm{H}_n(\mathrm{K}[\mathbf{x}';S]) \cong \begin{cases} S/(x+y) & \text{for } n = 0\\ 0 & \text{otherwise.}\end{cases}$$

Next we show that Betti or Bass numbers over φ are occasionally equal to the eponymous numbers over R, but may differ even when N is a finite R-module.

Remark 4.1.2. If the map $k \to S/\mathfrak{m}S$ induced by φ is bijective (for instance, if φ is the inclusion of R into its $\mathfrak{m}$-adic completion $\widehat{R}$, or if φ is surjective), then

$$\beta_n^{\varphi}(N) = \beta_n^{R}(N) \quad \text{and} \quad \mu_{\varphi}^{n}(N) = \mu_{R}^{n}(N).$$

Indeed, in this case the set $\mathbf{x}$ is empty, and ranks over l are equal to ranks over k.

Example 4.1.3. Set $S = R[[x]]$ and let $\varphi\colon R \to S$ be the canonical inclusion. The S-module $N = S/xS$ is finite over R, and elementary computations yield

$$\beta_n^{\varphi}(N) = \begin{cases} \beta_n^{R}(N) + 1 & \text{for } n = 1 \\ \beta_n^{R}(N) & \text{for } n \neq 1 \end{cases}$$

$$\mu_{\varphi}^{n}(N) = \mu_R^{n-1}(R) + \mu_R^{n}(R) \quad \text{for each} \quad n \in \mathbb{Z}.$$

In general, it is possible to express invariants over φ in terms of corresponding invariants over a ring, but a new ring is necessary:

Remark 4.1.4. If $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ is a minimal Cohen factorization of $\grave{\varphi}$, then for each $n \in \mathbb{Z}$ the following equalities hold:

$$\begin{aligned} \beta_n^{\varphi}(N) &= \beta_n^{\grave{\varphi}}(\widehat{N}) = \beta_n^{\widehat{\varphi}}(\widehat{N}) = \beta_n^{R'}(\widehat{N}) \\ \mu_{\varphi}^{n}(N) &= \mu_{\grave{\varphi}}^{n}(\widehat{N}) = \mu_{\widehat{\varphi}}^{n}(\widehat{N}) = \mu_{R'}^{n}(\widehat{N}). \end{aligned}$$

Indeed, the isomorphisms of S-modules in (4.1.0) yield $\beta_n^{\varphi}(N) = \beta_n^{R'}(\widehat{N})$. The same argument applied to $\grave{\varphi}$ yields $\beta_n^{\grave{\varphi}}(\widehat{N}) = \beta_n^{R'}(\widehat{N})$, and applied to $\widehat{\varphi}$ gives $\beta_n^{\widehat{\varphi}}(\widehat{N}) = \beta_n^{R'}(\widehat{N})$. Bass numbers are treated in a similar fashion.

It is often possible to reduce the study of sequences of Betti numbers to that of sequences of Bass numbers, and vice versa.

4.2. Duality. Let D be a dualizing complex for S, see (3.5), and set

$$N_S^{\dagger} = \mathbf{R}\mathrm{Hom}_S(N, D)\,.$$

LEMMA 4.2.1. *If S is complete and $R' \to S$ is a surjective homomorphism, then for each $n \in \mathbb{Z}$ there is an isomorphism of l-vector spaces*

$$\mathrm{Ext}_{R'}^{n}(l, N_S^{\dagger}) \cong \mathrm{Hom}_l(\mathrm{Tor}_n^{R'}(l, N), l).$$

Proof. The isomorphisms

$$\mathbf{R}\mathrm{Hom}_S((l \otimes^{\mathbf{L}}_{R'} N), D) \simeq \mathbf{R}\mathrm{Hom}_{R'}(l, \mathbf{R}\mathrm{Hom}_S(N, D)) = \mathbf{R}\mathrm{Hom}_{R'}(l, N^{\dagger}_S)$$

yield a spectral sequence with

$$E_2^{pq} = \mathrm{Ext}^p_S(\mathrm{Tor}^{R'}_q(l, N), D) \Rightarrow \mathrm{Ext}^{p+q}_{R'}(l, N^{\dagger}_S).$$

The S-module $\mathrm{Tor}^{R'}_q(l, N)$ is a direct sum of copies of l, so $E_2^{pq} = 0$ for $p \neq 0$ and $E^{0q} \cong \mathrm{Hom}_l(\mathrm{Tor}^{R'}_q(l, N), l)$ by the defining property of D; the assertion follows. □

THEOREM 4.2.2. *For each $n \in \mathbb{Z}$ the following equalities hold:*

$$\mu^n_{\varphi}(N) = \beta^{\grave{\varphi}}_n(\widehat{N}^{\dagger}_{\widehat{S}}) \qquad \textit{and} \qquad \beta^{\varphi}_n(N) = \mu^n_{\grave{\varphi}}(\widehat{N}^{\dagger}_{\widehat{S}}).$$

Proof. By Remark 4.1.4 we may assume that S is complete. As N and $N^{\dagger\dagger}_{SS}$ are isomorphic, it is enough to justify the second equality. If $R \to R' \to S$ is a minimal Cohen factorization of φ then the preceding lemma gives the middle equality below:

$$\mu^n_{\grave{\varphi}}(N^{\dagger}_S) = \mu^n_{R'}(N^{\dagger}_S) = \beta^{R'}_n(N) = \beta^{\varphi}_n(N).$$

The equality at each end is given by Remark 4.1.4. □

4.3. Poincaré series and Bass series. To study Betti numbers or Bass numbers we often use their generating functions. We call the formal Laurent series

$$P^{\varphi}_N(t) = \sum_{n \in \mathbb{Z}} \beta^{\varphi}_n(N) t^n \qquad \text{and} \qquad I^N_{\varphi}(t) = \sum_{n \in \mathbb{Z}} \mu^n_{\varphi}(N) t^n$$

the *Poincaré series* of N over φ and the *Bass series* of N over φ, respectively. When $\varphi = \mathrm{id}^R$, we speak of the Poincaré series and the Bass series of the complex N over the ring R, and write $P^R_N(t)$ and $I^N_R(t)$, respectively.

It is often convenient to work with sets of generators of $\mathfrak{n}$ modulo $\mathfrak{m}$ that are not necessarily minimal. The next results provides the necessary information.

PROPOSITION 4.3.1. *If $\mathbf{y} = \{y_1, \dots, y_q\}$ generates $\mathfrak{n}$ modulo $\mathfrak{m}S$, then*

$$\begin{aligned} P^{\varphi}_N(t) \cdot (1+t)^{q - \operatorname{edim}\varphi} &= \sum_{n \in \mathbb{Z}} \mathrm{rank}_l \left(\mathrm{Tor}^R_n(k, \mathrm{K}[\mathbf{y}; N])\right) t^n \\ I^N_{\varphi}(t) \cdot (1+t)^{q - \operatorname{edim}\varphi} &= \sum_{n \in \mathbb{Z}} \mathrm{rank}_l \left(\mathrm{Ext}^n_R(k, \mathrm{K}[\mathbf{y}; N])\right) t^{q+n}. \end{aligned}$$

Proof. The two formulas admit similar proofs. We present the first one.

To simplify notation, for any finite subset $\mathbf{z}$ of $\mathfrak{n}$ we set

$$F_{\mathbf{z},N}(t) = \sum_{n\in\mathbb{Z}} \ell_S\left(\operatorname{Tor}^R_n(k, \mathrm{K}[\mathbf{z}; N])\right) t^n.$$

Let $\mathbf{v} = \{v_1, \dots, v_r\}$ be a minimal set of generators of $\mathfrak{m}$. In $\mathcal{D}(S)$ one has

$$\mathrm{K}[\mathbf{y} \sqcup \varphi(\mathbf{v}); N] \simeq \mathrm{K}[\mathbf{y}; N] \otimes_S \mathrm{K}[\mathbf{v}; S] \simeq \mathrm{K}[\mathbf{y}; N] \otimes^{\mathbf{L}}_R \mathrm{K}[\mathbf{v}; R].$$

These isomorphisms induce the first link in the chain

$$\begin{aligned} k \otimes^{\mathbf{L}}_R (\mathrm{K}[\mathbf{y} \sqcup \varphi(\mathbf{v}); N]) &\simeq k \otimes^{\mathbf{L}}_R \left(\mathrm{K}[\mathbf{y}; N] \otimes^{\mathbf{L}}_R \mathrm{K}[\mathbf{v}; R]\right) \\ &\simeq \mathrm{K}[\mathbf{y}; N] \otimes^{\mathbf{L}}_R (k \otimes_R \mathrm{K}[\mathbf{v}; R]) \\ &\simeq \mathrm{K}[\mathbf{y}; N] \otimes^{\mathbf{L}}_R \left(\textstyle\bigwedge_k \Sigma k^r\right) \\ &\simeq \left(k \otimes^{\mathbf{L}}_R \mathrm{K}[\mathbf{y}; N]\right) \otimes_k \left(\textstyle\bigwedge_k \Sigma k^r\right). \end{aligned}$$

The third one holds because $\mathbf{v} \subseteq \mathfrak{m}$ implies that $k \otimes_R \mathrm{K}[\mathbf{v}; R]$ has trivial differential; the other two are standard. Taking homology and counting ranks over l we get

$$F_{\mathbf{y}\sqcup\varphi(\mathbf{v}),N}(t) = F_{\mathbf{y},N}(t) \cdot (1 + t)^r.$$

Let $\mathbf{x} = \{x_1, \dots, x_e\}$ be a minimal set of generators of $\mathfrak{n}$ modulo $\mathfrak{m}S$; clearly, one has $e \leq q$. Let $\mathbf{u}$ be generating set of $\mathfrak{m}$ consisting of $\mathbf{v}$ and $q - e$ additional elements, all equal to 0. The formula above then yields

$$F_{\mathbf{x}\sqcup\varphi(\mathbf{u}),N}(t) = F_{\mathbf{x},N}(t) \cdot (1 + t)^{r+q-e}.$$

Applying Lemma 3.1.3 to $\mathbf{z} = \mathbf{y} \sqcup \varphi(\mathbf{v})$ and to $\mathbf{z} = \mathbf{x} \sqcup \varphi(\mathbf{u})$ one gets $\mathrm{K}[\mathbf{y} \sqcup \varphi(\mathbf{v}); N] \cong \mathrm{K}[\mathbf{x} \sqcup \varphi(\mathbf{u}); N]$, so $F_{\mathbf{y}\sqcup\varphi(\mathbf{v}),N}(t) = F_{\mathbf{x}\sqcup\varphi(\mathbf{u}),N}(t)$. □

Additional elbow room is provided by the use of arbitrary Cohen factorizations.

PROPOSITION 4.3.2. *If $R \xrightarrow{\acute{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ is any Cohen factorization of $\acute{\varphi}$, then*

$$\begin{aligned} P^{\varphi}_N(t) \cdot (1 + t)^{\operatorname{edim}\acute{\varphi} - \operatorname{edim}\varphi} &= P^{R'}_{\widehat{N}}(t) \\ I^N_{\varphi}(t) \cdot (1 + t)^{\operatorname{edim}\acute{\varphi} - \operatorname{edim}\varphi} &= I^{\widehat{N}}_{R'}(t). \end{aligned}$$

Proof. By (4.1.4), we may assume that S is complete. In view of Theorem 4.2.2 and Lemma 4.2.1, it suffices to prove the expression for Poincaré series.

Let $\mathfrak{m}'$ be the maximal ideal of R'. Choose a set $\mathbf{y}$ that minimally generates $\mathfrak{m}'$ modulo $\mathfrak{m}R'$, set $q = \operatorname{card}\mathbf{y}$, and note that $\operatorname{edim}\dot{\varphi} = q$. The ring $P = R'/\mathfrak{m}R'$ is regular, so $\mathrm{K}[\mathbf{y}; P]$ is a resolution of l, hence $\mathrm{K}[\mathbf{y}; P] \simeq l$. From this isomorphism and the associativity of derived tensor products we get

$$\begin{aligned} l \otimes^{\mathbf{L}}_{R'} N &\simeq \mathrm{K}[\mathbf{y}; P] \otimes^{\mathbf{L}}_{R'} N \\ &\simeq P \otimes^{\mathbf{L}}_{R'} \mathrm{K}[\mathbf{y}; N] \\ &\simeq (k \otimes^{\mathbf{L}}_{R} R') \otimes^{\mathbf{L}}_{R'} \mathrm{K}[\mathbf{y}; N] \\ &\simeq k \otimes^{\mathbf{L}}_{R} \mathrm{K}[\mathbf{y}; N]. \end{aligned}$$

Passing to homology, we obtain the first equality below:

$$P^{R'}_N(t) = \sum_{n\in\mathbb{Z}} \operatorname{rank}_l \left(\operatorname{Tor}^R_n(k, \mathrm{K}[\mathbf{y}; N])\right) t^n = P^{\varphi}_N(t) \cdot (1+t)^{q-\operatorname{edim}\varphi}.$$

The second equality comes from Proposition 4.3.1. □

5. Illustrations. Betti numbers or Bass numbers over the identity map of R are equal to the corresponding numbers over the ring R, and decades of research have demonstrated that "computing" the latter invariants is in general a very difficult task. Nonetheless, drawing on techniques developed for the absolute case, or singling out interesting special classes of maps, it is sometimes possible to express invariants over φ in closed form, or to relate them to better understood entities. In this section we present several such results, some of which are used later in the paper.

5.1. Trivial actions. We record a version of the Künneth formula.

Remark 5.1.1. If V is a complex of S-modules satisfying $\mathfrak{m}V = 0$, then there exist isomorphisms of graded S-modules

$$\operatorname{Tor}^R(k, V) \cong \operatorname{Tor}^R(k, k) \otimes_k \mathrm{H}(V).$$

Indeed, V is naturally a complex over $S/\mathfrak{m}S$. If F is a free resolution of k over R, and $\overline{F} = F/\mathfrak{m}F$, then there are isomorphisms of complexes of S-modules

$$F \otimes_R V \cong \overline{F} \otimes_k V.$$

Expressing their homology via the Künneth formula, one gets the desired assertion.

Example 5.1.2. If $\mathfrak{m}N = 0$, then setting $s = \operatorname{edim} S$ one has

$$(1+t)^{s-\operatorname{edim}\varphi} \cdot P_N^{\varphi}(t) = P_k^R(t) \cdot K_N^S(t)$$
$$(1+t)^{s-\operatorname{edim}\varphi} \cdot I_{\varphi}^N(t) = P_k^R(t) \cdot t^s\, K_N^S(t^{-1}),$$

where $K_N^S(t)$ is the polynomial defined in (3.2). Remark 5.1.1, applied to the complex $V = \mathrm{K}^S[N]$ yields the expression for $P_N^{\varphi}(t)$. It implies the one for $I_{\varphi}^N(t)$, via Theorems 4.2.2 and 3.7.4.

5.2. Regular source rings. When one of the rings R or S is regular, homological invariants over the other ring dominate the behavior of the Betti numbers of N over φ and its Bass numbers over φ. Here we consider the case when the base ring is regular; the case of a regular target is dealt with in Corollary 6.2.3.

Proposition 5.2.1. *If the ring R is regular, then*

$$P_N^{\varphi}(t) = K_N^S(t) \cdot (1+t)^m \qquad \textit{and} \qquad I_{\varphi}^N(t) = K_N^S(t^{-1}) \cdot t^s\,(1+t)^m$$

where $m = \operatorname{edim} R - \operatorname{edim} S + \operatorname{edim}\varphi$ *and* $s = \operatorname{edim} S$.

Proof. The complex K^R is a free resolution of k over R, so $\operatorname{Tor}^R(k, \mathrm{K}^S[N]) \cong \mathrm{H}(K^R \otimes_R \mathrm{K}^S[N])$. This isomorphism gives the second equality below:

$$\begin{aligned} P_N^{\varphi}(t) \cdot (1+t)^{\operatorname{edim} S - \operatorname{edim}\varphi} &= \sum_{n\in\mathbb{Z}} \ell_S\left(\operatorname{Tor}_n^R(k, \mathrm{K}^S[N])\right) t^n \\ &= \sum_{n\in\mathbb{Z}} \ell_S\left(\mathrm{H}(K^R \otimes_R \mathrm{K}^S[N])\right) t^n \\ &= K_N^S(t) \cdot (1+t)^{\operatorname{edim} R}. \end{aligned}$$

The first one comes Proposition 4.3.1, the last from Lemma 3.1.3.

We have proved the expression for $P_N^{\varphi}(t)$. The one for $I_{\varphi}^N(t)$ is obtained from it, by applying Theorems 4.2.2 and 3.7.4. □

Remark 5.2.2. Recall that $\mathcal{D}^{\mathrm{f}}(S)$ is the derived category of homologically finite complexes of S-modules. If R is regular, then the proposition shows that $P_X^{\varphi}(t)$ and $I_{\varphi}^X(t)$ are Laurent polynomials for every $X \in \mathcal{D}^{\mathrm{f}}(S)$. Conversely, if $P_V^{\varphi}(t)$ or $I_{\varphi}^V(t)$ is a Laurent polynomial for some $V \in \mathcal{D}^{\mathrm{f}}(S)$ with $\mathrm{H}(V) \neq 0$ and $\mathfrak{m}V = 0$, then Remark 5.1.2 implies that $P_k^R(t)$ is a polynomial, hence that R is regular.

5.3. Complete intersection source rings. We start with some terminology.

5.3.1. The ring R is *complete intersection* if in some Cohen presentation $\widehat{R} \cong Q/\mathfrak{a}$, see (3.4), the ideal $\mathfrak{a}$ can be generated by a regular set. When this is the case, the defining ideal in each (minimal) Cohen presentation of $\widehat{R}$ is generated by a regular set (of cardinality $\operatorname{codim} R$). Complete intersection rings of codimension at most 1 are also called *hypersurface rings*.

In the next result we use the convention $0! = 1$.

THEOREM 5.3.2. *Assume R is complete intersection and set $c = \operatorname{codim} R$. When* flat $\dim_R N = \infty$ *there exist polynomials $b_{\pm}(x) \in \mathbb{Q}[x]$ of the form*

$$b_{\pm}(x) = \frac{b_N}{2^c(d-1)!} \cdot x^{d-1} + \textit{ lower order terms}$$

with an integer $b_N > 0$ and $1 \le d \le c$, such that

$$\beta_n^{\varphi}(N) = \begin{cases} b_+(n) & \textit{for all even } n \gg 0; \\ b_-(n) & \textit{for all odd } n \gg 0. \end{cases}$$

In particular, if $d = 1$, then $\beta_n^{\varphi}(N) = \beta_{n-1}^{\varphi}(N)$ for all $n \gg 0$.

If $d \ge 2$, then there exist polynomials $a_{\pm}(x) \in \mathbb{Q}[x]$ of degree $d-2$ with positive leading coefficients, such that

$$\beta_{n-1}^{\varphi}(N) + a_+(n) \ge \beta_n^{\varphi}(N) \ge \beta_{n-1}^{\varphi}(N) + a_-(n) \quad \textit{for all} \quad n \gg 0.$$

The corresponding assertions for the Bass numbers $\mu_{\varphi}^n(N)$ hold as well.

Proof. Let $R \to R' \to \widehat{S}$ be a minimal Cohen factorization of $\grave{\varphi}$. Choose an isomorphism $F' \simeq \widehat{N}$, where F' is a complex of finite free R'-modules with $F'_n = 0$ for all $n \le i$, where $i = \inf \mathrm{H}(N)$. Set $L = \partial(F_s)$, where $s = \sup \mathrm{H}(N)$. The complex $F_{i \geqslant s}$ is a free resolution of L over R', hence the second equality below:

$$\beta_{j+s}^{\varphi}(N) = \beta_{j+s}^{R'}(\widehat{N}) = \beta_j^{R'}(L) \quad \text{for all} \quad j > 0.$$

The first equality comes from Remark 4.1.4.

As R' is flat over the complete intersection ring R, and the ring $R'/\mathfrak{m}R'$ is regular, R' is complete intersection with $\operatorname{codim} R' = c$. The properties of the Betti numbers of L over R, which translate into the desired properties of the Betti numbers of N over φ, are given by [11, (8.1)]. That the Bass numbers of M have the corresponding properties is now seen from Theorem 4.2.2. $\square$

5.4. Weakly regular homomorphisms. Let $\psi\colon (Q, \mathfrak{l}, h) \to (R, \mathfrak{m}, k)$ be a local homomorphism.

5.4.1. We say that ψ is *weakly regular* at $\mathfrak{m}$ if the map $\grave{\psi}$ has a Cohen factorization $Q \to (Q', \mathfrak{l}', k) \xrightarrow{\psi'} \widehat{R}$ with $\operatorname{Ker} \psi'$ generated by a *superficial regular set* $\mathbf{f}'$, that is, one whose image in $\mathfrak{l}'/\mathfrak{l}'^2$ spans a k-subspace of rank $\operatorname{card} \mathbf{f}'$. If

ψ is weakly regular at $\mathfrak{m}$, then in each Cohen factorization of $\grave{\psi}$ the kernel of the surjection is generated by a superficial regular set: this follows from the Comparison Theorem [10, (1.2)].

THEOREM 5.4.2. *If ψ is weakly regular, then*

$$P_N^{\varphi}(t) \cdot (1+t)^{\dim Q - \dim R} = P_N^{\varphi\circ\psi}(t) \cdot (1+t)^{\operatorname{edim}\varphi - \operatorname{edim}(\varphi\circ\psi)}.$$

A similar equality holds for the Bass series of N.

One ingredient of the proof is the following lemma, which extends [6, (3.3.5)].

LEMMA 5.4.3. *If $f \in \mathfrak{l} \setminus \mathfrak{l}^2$ is a regular element and $R = Q/(f)$, then*

$$P_N^{\varphi\circ\psi}(t) = P_N^{\varphi}(t) \cdot (1+t)$$

where $\psi\colon\ Q \to R$ is the canonical map.

Proof. We start by fixing some notation. Let $\mathbf{x}$ be a minimal set of generators of $\mathfrak{n}$ modulo $\mathfrak{m}S$; it also minimally generates $\mathfrak{n}$ modulo $\mathfrak{l}S$, since ψ is surjective.

Let $\mathrm{K}[f;Q]$ be the Koszul complex of f, and let E denote the underlying exterior algebra. Since $f \notin \mathfrak{l}^2$, one can extend $\mathrm{K}[f;Q]$ to an acyclic closure Y of k over Q; see [6, (6.3.1)]. Moreover, there exists a derivation θ on Y, compatible with its divided powers structure, such that $\theta(e_f) = 1$, see [6, (6.3.3)]. The induced derivation $\theta\otimes R$ on the DG algebra $Y\otimes_Q R$ satisfies $(\theta\otimes R)(e_f\otimes 1) = 1$; in addition, $\partial(e_f \otimes 1) = 0$. Under these conditions, André [1, Proposition 6] shows that there exists a complex of R-modules X that appears in an isomorphism of complexes of R-modules

$$Y \otimes_Q R \cong X \oplus \Sigma X.$$

Note that $\mathrm{H}(Y \otimes_Q R) \cong \operatorname{Tor}^Q(k,R) \cong k \oplus \Sigma k$, so $\mathrm{H}(X) \cong k$. In addition, since Y is complex of free Q-modules, the complex of R-modules X is free. Thus, X is a free resolution of k over R. Tensoring the isomorphism above with $\mathrm{K}[\mathbf{x};N]$ yields

$$\begin{aligned} Y \otimes_Q \mathrm{K}[\mathbf{x};N] &= (Y \otimes_Q R) \otimes_R \mathrm{K}[\mathbf{x};N] \\ &\cong (X \otimes_R \mathrm{K}[\mathbf{x};N]) \oplus \Sigma\,(X \otimes_R \mathrm{K}[\mathbf{x};N])\,. \end{aligned}$$

Taking homology one obtains an isomorphism of graded k-vector spaces

$$\operatorname{Tor}^Q(k,\mathrm{K}[\mathbf{x};N]) \cong \operatorname{Tor}^R(k,\mathrm{K}[\mathbf{x};N]) \oplus \Sigma\operatorname{Tor}^R(k,\mathrm{K}[\mathbf{x};N])$$

which yields the desired equality of Poincaré series. □

Also needed for the theorem is a construction from [9, (4.4)], see also [20, (5.9)].

Construction 5.4.4. Assume that the rings R and S are complete.

Let $Q \xrightarrow{\dot\psi} Q' \xrightarrow{\psi'} R$ and $Q' \xrightarrow{\ddot\varkappa} Q'' \xrightarrow{\varkappa'} S$ be minimal Cohen factorizations of ψ and $\varphi\psi'$, respectively. The ring $R' = R \otimes_{Q'} Q''$ and the maps

$$\psi'' = \psi' \otimes_{Q'} Q'' , \qquad \dot\varphi = R \otimes_{Q'} \ddot\varkappa , \qquad \dot\varkappa = \ddot\varkappa\dot\psi ,$$

fit into a commutative diagram of local homomorphisms

$$\begin{tikzcd}
 & & Q'' \arrow[dr, "\psi''"] \arrow[rrdd, bend left, "\varkappa'"] & & \\
 & Q' \arrow[ur, "\ddot\varkappa"] \arrow[dr, "\psi'"] & & R' \arrow[dr, "\varphi'"] & \\
Q \arrow[ur, "\dot\psi"] \arrow[rr, "\psi"] \arrow[rruu, bend left, "\dot\varkappa"] & & R \arrow[ur, "\dot\varphi"] \arrow[rr, "\varphi"] & & S
\end{tikzcd}$$

where φ' is the induced map. It contains the following Cohen factorizations:

(1) $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} S$ of φ, which is minimal;

(2) $Q \xrightarrow{\dot\varkappa} Q'' \xrightarrow{\varkappa'} S$ of $\varphi\psi$, in which $\operatorname{edim}\dot\varkappa = \operatorname{edim}\varphi + \operatorname{edim}\psi$.

Proof of Theorem 5.4.2. By Remark 4.1.4 we may assume R and S complete. We form the diagram of the construction above and adopt its notation.

Since ψ is weakly regular at $\mathfrak{m}$, the kernel of ψ' is generated by a superficial regular set $\mathbf{f}'$, see (5.4.1). Since $\psi'' = Q'' \otimes_{Q'} \psi'$, and Q'' is faithfully flat over Q', the image $\mathbf{f}''$ of $\mathbf{f}'$ in Q'' is a regular set of card $\mathbf{f}'$ elements, and generates the kernel of ψ''. Let $\mathfrak{l}'$ and $\mathfrak{l}''$ be the maximal ideals of Q' and Q'', respectively. As $Q''/\mathfrak{l}'Q''$ is regular, any minimal set of generators of the ideal $\mathfrak{l}'Q'' = \operatorname{Ker}(Q'' \to Q''/\mathfrak{l}'Q'')$ extends to a minimal set of generators of $\mathfrak{l}''$. It follows that $\mathbf{f}''$ is superficial.

The first equality below holds because $\varkappa'$ is surjective, see (4.1.2), the second from iterated applications of Lemma 5.4.3:

$$P_N^{Q''}(t) = P_N^{\varkappa'}(t) = P_N^{\varphi'}(t) \cdot (1+t)^{\dim Q - \dim R + \operatorname{edim}\psi} .$$

Property (2) of Construction 5.4.4 shows that $\varkappa'\dot\varkappa$ is a Cohen factorization of $\varphi\,\psi$, with $\operatorname{edim}\dot\varkappa = \operatorname{edim}\varphi + \operatorname{edim}\psi$. Thus, from Proposition 4.3.2 we get

$$\begin{aligned} P_N^{Q''}(t) &= P_N^{\varphi\,\psi}(t) \cdot (1+t)^{\operatorname{edim}\dot\varkappa - \operatorname{edim}(\varphi\,\psi)} \\ &= P_N^{\varphi\,\psi}(t) \cdot (1+t)^{\operatorname{edim}\varphi + \operatorname{edim}\psi - \operatorname{edim}(\varphi\,\psi)} . \end{aligned}$$

Comparison of the two expressions for $P_N^{Q''}(t)$ finishes the proof for Poincaré series.

The equality for Bass series is a formal consequence, due to Theorem 4.2.2. □

6. Separation. This section introduces one of the main themes of the paper—the comparison of homological invariants of N over φ with homological invariants of k over R.

We start by establishing upper bounds on the Poincaré series and on the Bass series of N. The bounds involve series with specific structures: input from each of the rings R and S and from the map φ appear as separate factors. We say that N is (injectively) separated if the upper bound is reached, and we turn to the natural problem of identifying cases when this happens. Several sufficient conditions are obtained, in terms of the structure of N over R, the properties of the ring R, those of the ring S, or the way that $\varphi(\mathfrak{m})$ sits inside $\mathfrak{n}$. Finally, we obtain detailed information on Betti sequences and Bass sequences of (injectively) separated complexes. To do this, we use the specific form of their Poincaré series or Bass series along with extensive information on $P_k^R(t)$, available from earlier investigations.

6.1. Upper bound. In the theorem below $K_N^S(t)$ denotes the Laurent polynomial defined in (3.2). The first inequality and its proof are closely related to results of Lescot in [24, §2]; see Remark 6.3.5.

The symbols $\preccurlyeq$ and $\succcurlyeq$ denote coefficientwise inequalities of formal Laurent series.

THEOREM 6.1.1. *Assume* $\mathrm{H}(N) \neq 0$, *and set* $s = \operatorname{edim} S$ *and* $e = \operatorname{edim} \varphi$.
There is an inequality of formal Laurent series

$$(1+t)^{s-e} \cdot P_N^{\varphi}(t) \preccurlyeq P_k^R(t) \cdot K_N^S(t)$$

and on both sides the initial term is $(\nu_S \mathrm{H}_i(N))\, t^i$, *where* $i = \inf \mathrm{H}(N)$.
There is an inequality of formal Laurent series

$$(1+t)^{s-e} \cdot I_{\varphi}^N(t) \preccurlyeq P_k^R(t) \cdot K_N^S(t^{-1})\, t^s$$

and on both sides the initial term is $(\operatorname{type}_S N)\, t^g$, *where* $g = \operatorname{depth}_S N$.

Proof. Let F be a free resolution of k over R. Filter the complex $F \otimes_R \mathrm{K}^S[N]$ by its subcomplexes $F \otimes_R \left(\mathrm{K}^S[N]_{\leqslant q}\right)$ to get a spectral sequence

$$E^2_{pq} = \operatorname{Tor}_p^R\left(k, \mathrm{H}_q \mathrm{K}^S[N]\right) \Rightarrow \operatorname{Tor}_{p+q}^R\left(k, \mathrm{K}^S[N]\right).$$

The R-module $\mathrm{H}_q\mathrm{K}^S[N]$ is annihilated by $\mathfrak{m}$, so one has an isomorphism

$$\mathrm{Tor}_p^R\left(k, \mathrm{H}_q\mathrm{K}^S[N]\right) \cong \mathrm{Tor}_p^R(k,k) \otimes_k \mathrm{H}_q\mathrm{K}^S[N].$$

It is responsible for the last equality below

$$\begin{aligned} P_N^{\varphi}(t) \cdot (1+t)^{s-e} &= \sum_{n\in\mathbb{Z}} \mathrm{rank}_l \, \mathrm{Tor}_{p+q}^R\left(k, \mathrm{K}^S[N]\right) t^n \\ &\preccurlyeq \sum_{n\in\mathbb{Z}} \left(\sum_{p+q=n} \mathrm{rank}_l \, E_{p,q}^2 \right) t^n \\ &= P_k^R(t) \cdot K_N^S(t) \end{aligned}$$

while the first equality comes from Lemma 4.3.1. The convergence of the spectral sequence yields the inequality.

To identify the leading terms, note that Theorem 3.7.1 yields $\mathrm{H}_q(K^S[N]) = 0$ for $q < i$. Thus, in the spectral sequence above $E_{pq}^2 = 0$ when $p < 0$ or $q < i$, hence

$$\mathrm{Tor}_n^R\left(k, \mathrm{K}^S[N]\right) \cong \begin{cases} 0 & \text{for } n < i \\ \mathrm{H}_i(\mathrm{K}^S[N]) & \text{for } n = i. \end{cases}$$

This shows that $\mathrm{rank}_l \, \mathrm{H}_i(\mathrm{K}^S[N])\, t^i$ is the initial term of both series under consideration, and its coefficient is identified by Theorem 3.7.1.

The inequality concerning the Bass series of N and the equality of initial terms follow from the corresponding statements about the Poincaré series of $N^{\dagger}$, in view of the equalities $I_{\varphi}^N(t) = P_{N^{\dagger}}^{\varphi}(t)$ from Theorem 4.2.2, and $K_{N^{\dagger}}^S(t) = t^s K_N^S(t^{-1})$ from Theorem 3.7.4. To obtain the leading term itself use Theorem 3.7.2. □

The theorem gives a reason to consider the following classes of complexes.

6.1.2. We say that N is *separated* over φ if

$$P_N^{\varphi}(t) = P_k^R(t) \cdot \frac{K_N^S(t)}{(1+t)^{\operatorname{edim} S}} \cdot (1+t)^{\operatorname{edim}\varphi}.$$

Separation is related to Lescot's [24] notion of inertness, see Remark 6.3.5.

We say that N is *injectively separated* over φ if

$$I_{\varphi}^N(t) = P_k^R(t) \cdot \frac{K_N^S(t^{-1})\, t^{\operatorname{edim} S}}{(1+t)^{\operatorname{edim} S}} \cdot (1+t)^{\operatorname{edim}\varphi}.$$

Comparing the formulas above with those in Remark 5.1.2 and Proposition 5.2.1, we exhibit the first instances of separation.

Example 6.1.3. If $\mathfrak{m}N = 0$, or if the ring R is regular, then N is separated and injectively separated over φ.

To obtain further examples we introduce a concept of independent interest.

6.2. Loewy lengths. Let X be a complex of S-modules. The number

$$\ell\ell_S(X) = \inf\{i \in \mathbb{N} \mid \mathfrak{n}^i \cdot X = 0\}$$

is the *Loewy length* of X over S. To obtain an invariant in $\mathcal{D}(S)$, we introduce

$$\ell\ell_{\mathcal{D}(S)}(X) = \inf\{\ell\ell_S(V) \mid V \in \mathcal{D}(S) \text{ with } V \simeq X\}$$

as the *homotopical Loewy length* of X over S. Evidently, there are inequalities

$$\ell\ell_S(\mathrm{H}(X)) \leq \ell\ell_{\mathcal{D}(S)}(X) \leq \ell\ell_S(X).$$

Equalities hold when X is an S-module, but not in general; see Corollary 6.2.3.

LEMMA 6.2.1. *If X and Y are complexes of S-modules, then*

$$\ell\ell_{\mathcal{D}(S)}(X \otimes^{\mathbf{L}}_S Y) \leq \min\{\ell\ell_{\mathcal{D}(S)}(X), \ell\ell_{\mathcal{D}(S)}(Y)\}$$
$$\ell\ell_{\mathcal{D}(S)}(\mathbf{R}\mathrm{Hom}_S(X, Y)) \leq \min\{\ell\ell_{\mathcal{D}(S)}(X), \ell\ell_{\mathcal{D}(S)}(Y)\}.$$

Proof. We may assume $\ell\ell_{\mathcal{D}(S)}(X) = i < \infty$, hence $X \simeq V$ and $\mathfrak{n}^i \cdot V = 0$. Replace Y with a K-projective resolution F to get $X \otimes^{\mathbf{L}}_S Y \simeq V \otimes_S F$ and $\mathfrak{n}^i \cdot (V \otimes_S F) = 0$. This gives $\ell\ell_{\mathcal{D}(S)}(X \otimes^{\mathbf{L}}_S Y) \leq \ell\ell_{\mathcal{D}(S)}(X)$. The inequality $\ell\ell_{\mathcal{D}(S)}(X \otimes^{\mathbf{L}}_S Y) \leq \ell\ell_{\mathcal{D}(S)}(Y)$ follows by symmetry. A similar argument yields the second set of inequalities. □

Our interest in homotopical Loewy lengths is due to the next result, which also involves the invariant spread S introduced in (3.8).

THEOREM 6.2.2. *The following inequalities hold:*

$$\ell\ell_{\mathcal{D}(S)}(\mathrm{K}^S[N]) \leq \ell\ell_{\mathcal{D}(S)}(K^S) \leq \operatorname{spread} S.$$

The complex N is separated and injectively separated over φ if

$$\varphi(\mathfrak{m}) \subseteq \mathfrak{n}^q \quad \textit{where} \quad q = \ell\ell_{\mathcal{D}(S)}(\mathrm{K}^S[N]).$$

Proof. Lemma 6.2.1 gives the first one of the desired inequalites. On the other hand, Proposition 3.11 shows that, for $a = \operatorname{spread} S$, the complex

$$I^a = \quad 0 \longrightarrow \mathfrak{n}^{a-s}K^S_s \longrightarrow \mathfrak{n}^{a-s+1}K^S_{s-1} \longrightarrow \cdots \longrightarrow \mathfrak{n}^{a-1}K^S_1 \to \mathfrak{n}^a K^S_0 \to 0$$

is exact. Thus, $K^S \simeq K^S/I^a$; note that $\mathfrak{n}^a \cdot (K^S/I^a) = 0$, and so $q \leq a$.

Assume now $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}^q$. Let V be a complex of S-modules with $\mathfrak{n}^q V = 0$ and $V \simeq \mathrm{K}^S[N]$ in $\mathcal{D}(S)$. This explains the first and the last isomorphisms below

$$\begin{aligned} \operatorname{Tor}^R(k, \mathrm{K}^S[N]) &\cong \operatorname{Tor}^R(k, V) \\ &\cong \operatorname{Tor}^R(k,k) \otimes_k \mathrm{H}(V) \\ &\cong \operatorname{Tor}^R(k,k) \otimes_k \mathrm{H}(\mathrm{K}^S[N]). \end{aligned}$$

The second is Remark 5.1.1. Computing ranks and using Proposition 4.3.1 one gets separation over φ. A similar argument yields injective separation. □

As a corollary, we complement Proposition 5.2.1, see also Example 6.1.3.

COROLLARY 6.2.3. *When S is regular N is separated and injectively separated over φ. Also, S is regular if and only if $\ell\ell_{\mathcal{D}(S)}(K^S) \leq 1$, if and only if* $\operatorname{spread} S \leq 1$.

Proof. Assume first $\ell\ell_{\mathcal{D}(S)}(K^S) \leq 1$, that is, $K^S \simeq V$ for a complex of S-modules V modules with $\mathfrak{n} \cdot V = 0$. Thus, V is a complex of l-vector spaces, hence $\mathrm{H}_0(V) = l$ is isomorphic in $\mathcal{D}(S)$ to a direct summand of V. One sees that S is regular from:

$$\operatorname{flat\,dim}_S l \leq \operatorname{flat\,dim}_S V = \operatorname{flat\,dim}_S K^S = \operatorname{edim} S < \infty.$$

Assume next S is regular. In this case $\operatorname{gr}_{\mathfrak{n}}(S)$ is a polynomial algebra, so $\mathrm{H}(K^{\operatorname{gr}_{\mathfrak{n}}(S)}) \cong l$, and hence $\operatorname{spread} S \leq 1$ by Proposition 3.11. Now Theorem 6.2.2 yields an inequality $\ell\ell_{\mathcal{D}(S)}(K^S) \leq 1$ and the assertions about N. □

6.3. Properties. The first nonzero Betti number and the first nonzero Bass number of N are computed in Theorem 6.1.1. For separated complexes detailed information is also available on subsequent numbers in each sequence.

THEOREM 6.3.1. *Assume* $\mathrm{H}(N) \neq 0$, *and set* $i = \inf \mathrm{H}(N)$ *and* $m = \nu_S \mathrm{H}_i(N)$.
If N is separated over φ, then its Betti numbers have the following properties.
(1) *When R is not a hypersurface ring there are inequalities*

$$\beta^\varphi_n(N) \geq \begin{cases} \beta^\varphi_i(N) & \text{for } n = i+1 \\ \beta^\varphi_{n-1}(N) + m & \text{for } n \geq i+2. \end{cases}$$

If $\operatorname{codim} S \geq 2$ *or* $\operatorname{edim} R + \operatorname{edim} \varphi > \operatorname{edim} S$, *then also*

$$\beta^{\varphi}_{i+1}(N) \geq \beta^{\varphi}_{i}(N) + m.$$

(2) *When R is not complete intersection there is a real number* $b > 1$ *such that*

$$\beta^{\varphi}_{n}(N) \geq m\, b^{n-i} \quad \text{for all} \quad n \geq i + 2.$$

If N is injectively separated over φ*, then its Bass numbers over* φ *satisfy similar inequalities, obtained from those above by replacing* $\beta^{\varphi}_n(N)$ *with* $\mu^n_{\varphi}(N)$*, the number* $i = \inf \mathrm{H}(N)$ *with* $\operatorname{depth}_S N$*, and* $m = \nu_S \mathrm{H}_i(N)$ *with* $\operatorname{type}_S N$.

The theorem is proved at the end of this section.

Remark 6.3.2. An inequality $\operatorname{edim} R + \operatorname{edim} \varphi \geq \operatorname{edim} S$ always holds, and there is equality if and only if some (respectively, each) minimal Cohen factorization $R \to R' \to \widehat{S}$ of $\acute{\varphi}$ satisfies $\operatorname{edim} R' = \operatorname{edim} S$.

Indeed, if $\mathbf{v}$ minimally generates $\mathfrak{m}$ and $\mathbf{x}$ minimally generates $\mathfrak{n}$ modulo $\mathfrak{m}S$, then $\varphi(\mathbf{v}) \cup \mathbf{x}$ generates $\mathfrak{n}$, hence the inequality. In a minimal Cohen factorization $\operatorname{edim} R' = \operatorname{edim} R + \operatorname{edim} \varphi$, so equality holds if and only if $\operatorname{edim} R' = \operatorname{edim} S$.

Next we show that the results in the theorem are optimal.

Example 6.3.3. Let k be a field and set $R = k[[x, y]]/(x^2, xy)$. One then has

$$P^R_k(t) = \frac{(1 + t)^2}{1 - 2t^2 - t^3} = \frac{1 + t}{1 - t - t^2}$$

see [6, (5.3.4)]. Set $S = R$ and let N be the S-module $S/(x)$. Set $\varphi = \mathrm{id}^R$, so $P^{\varphi}_N(t) = P^R_N(t)$. The exact sequence $0 \to k \to R \to N \to 0$ yields

$$P^R_N(t) = 1 + tP^R_k(t) = \frac{1}{1 - t - t^2}$$

Furthermore, one has $K^S_N(t) = 1 + t$, e.g. from Theorem 3.7.3 with $T = k[[x, y]]$.

The computations above show that N is separated. However $\beta^{\varphi}_1(N) = \beta^{\varphi}_0(N) = 1$, so equality holds in (1), and the inequality in (2) fails for $n = i + 1$.

Remark 6.3.4. If $R \xrightarrow{\acute{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ is a minimal Cohen factorization of $\widehat{S}$, then N is separated over φ if and only if $\widehat{N}$ is separated over φ'.

Indeed, set $e = \operatorname{edim} \varphi$. As $\acute{\varphi}$ is flat and $R'/\mathfrak{m}R'$ is regular, it is not hard to show $P^{R'}_l(t) = P^R_k(t)(1 + t)^e$. The equality $P^{\varphi}_N(t) = P^{\varphi'}_{\widehat{N}}(t)$ yields our assertion.

We are ready to compare separation with inertness, as defined by Lescot [24].

Remark 6.3.5. An S-module L is *inert* if $(1+t)^{\operatorname{edim} S} \cdot P_L^S(t) = P_k^S(t) \cdot K_L^S(t)$, see [24, (2.5)]. This is precisely the condition that L is separated over id^S.

Suppose $\operatorname{edim} S = \operatorname{edim} R + \operatorname{edim} \varphi$. Remark 6.3.2 shows that $\grave{\varphi}$ has a minimal Cohen factorization with $\operatorname{edim} R' = \operatorname{edim} S$, so $K_L^S(t) = K_L^{R'}(t)$. By the preceding remark, L is separated over φ if and only if it is inert over R'.

When φ induces the identity on k and the ring $S/\mathfrak{m}S$ is artinian, Lescot says that L is *inert through* φ if $P_k^S(t) \cdot P_L^R(t) = P_L^S(t) \cdot P_k^R(t)$. This condition is very different from separation over φ: it can be shown that $L \neq 0$ has both properties if and only if it is inert over R, the map φ is flat, and $\mathfrak{n} = \mathfrak{m}S$.

Next we recall background information on the homology of k.

6.3.6. Set $r = \operatorname{edim} R$, and let p denote minimal number of generators of the ideal $\mathfrak{a}$ in a minimal Cohen presentation $\widehat{R} \cong Q/\mathfrak{a}$. The series $P_k^R(t)$ can be written as

$$P_k^R(t) = \frac{(1+t)^r}{(1-t^2)^p} \cdot F(t) \quad \text{where} \quad F(t) = \frac{\prod_{i=1}^{\infty} (1+t^{2i+1})^{\varepsilon_{2i+1}(R)}}{\prod_{i=1}^{\infty} (1-t^{2i+2})^{\varepsilon_{2i+2}(R)}}$$

for uniquely determined integers $\varepsilon_n(R) \geq 0$ for $n \geq 3$, see [6, (7.1.4)].

(1) The ring R is regular if and only $p = 0$; in this case $F(t) = 1$.

(2) The ring R is complete intersection if and only $\varepsilon_3(R) = 0$, if and only if $F(t) = 1$.

(3) If R is not complete intersection, then there exists an integer $s_1 \geq 2$ and a sequence of integers $(i_j)_{j=1}^{\infty}$ with $2 \leq i_j \leq \operatorname{edim} R + 1$ for all j, such that

$$s_{j+1} = i_j(s_j - 1) + 2 \quad \text{satisfy} \quad \varepsilon_{s_j}(R) \geq a^{s_j}$$

for some real number $a > 1$.

Of the assertions above (1) is clear, (2) is due to Assmus and Tate, see [6, (7.3.3)], and (3) is due to Avramov, see [6, (8.2.3)].

To prove the next lemma we abstract an argument from the proof of [6, (8.2.1)].

LEMMA 6.3.7. *Assume R is not a complete intersection and let $F(t)$ be as above.*

(1) *The following inequalities hold:*

$$\frac{F(t)}{(1-t^2)} \succcurlyeq \frac{1+t^3}{(1-t^2)} = 1 + \sum_{h=2}^{\infty} t^h.$$

(2) *There exists a real number $b > 1$, such that the following inequality holds:*

$$\frac{F(t)}{(1-t)(1-t^2)} \succcurlyeq 1 + t + \sum_{h=2}^{\infty} b^h t^h.$$

Proof. (1) is clear from (6.3.6.2).

(2) Set $\sum_{h\in\mathbb{Z}} a_h t^h = F(t)/(1-t^2)$ and $\sum_{h\in\mathbb{Z}} b_h t^h = F(t)/((1-t)(1-t^2))$. Since $b_h = a_h + \cdots + a_0$ for each $h \geq 0$, we get $b_1 \geq b_0 \geq 1$ and

$$b_{h+1} = \left(\sum_{j=0}^{h+1} a_j\right) > \left(\sum_{j=0}^{h} a_j\right) = b_h \qquad \text{for all} \qquad h \geq 1.$$

Let $s_1, s_2, \ldots$ be the numbers from (6.3.6.3) and set $r = \operatorname{edim} R$. The number

$$b = \min\left\{ \sqrt[r+1]{a}\,,\ \sqrt{b_2}\,, \ldots,\ \sqrt[s_1]{b_{s_1}} \right\}$$

satisfies $b_h \geq b^h > 1$ for $s_1 \geq h \geq 2$. If $s_{j+1} \geq h > s_j$ with $j \geq 1$, then

$$b_h > b_{s_j} \geq \varepsilon_{s_j}(R) \geq a^{s_j} \geq b^{(r+1)s_j} > b^{s_{j+1}} > b^h$$

so we see that $b_h \geq b^h$ holds for all $h \geq 2$. This is the desired inequality. □

Proof of Theorem 6.3.1. We adopt the notation of Lemma 6.3.7, set

$$d = \operatorname{edim} R - \operatorname{edim} S + \operatorname{edim} \varphi.$$

Remark 6.3.2 and the hypothesis that R is not a hypersurface yield

(*) $$d \geq 0 \qquad \text{and} \qquad p \geq 2.$$

Theorem 3.7.7 provides a Laurent polynomial $L(t) \succcurlyeq m\,t^i$, such that

(†) $$K_N^S(t)(1+t)^d = L(t)(1+t)^{d+1}.$$

(1) From the discussion above we obtain the relation

$$\sum_{n=i}^{\infty} \left(\beta_n^{\varphi}(N) - \beta_{n-1}^{\varphi}(N)\right) t^n = (1-t)\cdot P_N^{\varphi}(t) = \frac{L(t)(1+t)^d}{(1-t^2)^{p-1}} \cdot F(t).$$

If $\operatorname{codim} S \geq 2$, then Theorem 3.7.5 yields $L(t) \succcurlyeq m\,t^i\,(1+t)$, in particular,

$$L(t)(1+t)^d \succcurlyeq m\,t^i\,(1+t).$$

The same inequality holds when $d \geq 1$. Thus, in these cases we get

$$\sum_{n=i}^{\infty} \left(\beta_n^{\varphi}(N) - \beta_{n-1}^{\varphi}(N)\right) t^n \;=\; \frac{L(t)(1+t)^d}{(1-t^2)} \cdot \frac{F(t)}{(1-t^2)^{p-2}}$$

$$\succcurlyeq \frac{m\, t^i\,(1+t)}{(1-t^2)}$$
$$= m\sum_{n=i}^{\infty} t^n.$$

This implies $\beta_n^{\varphi}(N) \geq \beta_{n-1}^{\varphi}(N) + m$ for all $n \geq i+1$, as desired.

Now we assume $\operatorname{codim} S = 1$ and $d = 0$. By Remark 6.3.2, $\acute{\varphi}$ has a minimal Cohen factorization $R \to R' \to \widehat{S}$ with $\operatorname{edim} R' = \operatorname{edim} S$, hence

$$\operatorname{codim} R = \operatorname{codim} R' = \operatorname{edim} R' - \dim R' \leq \operatorname{edim} S - \dim S = \operatorname{codim} S = 1.$$

Since R is not a hypersurface by hypothesis, the inequality above bars it from being complete intersection. Using Theorem 3.7.1 and Lemma 6.3.7.1 we obtain

$$\sum_{n=i}^{\infty} (\beta_n^{\varphi}(N) - \beta_{n-1}^{\varphi}(N))\, t^n = \frac{L(t)}{(1-t^2)^{p-2}} \cdot \frac{F(t)}{(1-t^2)} \succcurlyeq m\, t^i \cdot \left(1 + \sum_{h=2}^{\infty} t^h\right).$$

This yields $\beta_{i+1}^{\varphi}(N) \geq \beta_i^{\varphi}(N)$ and $\beta_n^{\varphi}(N) > \beta_{n-1}^{\varphi}(N)) + m$ for all $n \geq i+2$.

(2) Formula (†) gives the equality below:

$$P_N^{\varphi}(t) = L(t) \cdot \frac{(1+t)^d}{(1-t^2)^{p-2}} \cdot \frac{F(t)}{(1-t)(1-t^2)}$$
$$\succcurlyeq m\, t^i \cdot \left(1 + t + \sum_{h=2}^{\infty} b^h\, t^h\right).$$

The inequality, involving a real number $b > 1$, comes from Theorem 3.7.7, formula $(*)$, and Lemma 6.3.7.2. Thus, $\beta_n^{\varphi}(N) \geq m\, b^{n-i}$ holds for all $n \geq i+2$. □

7. Asymptotes. The data encoded in the sequences of Betti numbers and of Bass numbers of N are often too detailed to decipher. Many results suggest that it is the *asymptotic behavior* of these sequence that carries an understandable algebraic significance. In this section we introduce and initiate the study of a pair of numerical invariants that evaluate the (co)homological nature of N over R by measuring the asymptotic growth of the sequences of its Betti numbers and of its Bass numbers over φ.

We use two scales to measure the rate at which these numbers grow: a polynomial one, leading to the notion of complexity, and an exponential one, yielding that of curvature. One reason for restricting to these scales is that these numbers grow at most exponentially, and there is no example with rate of growth intermediate between polynomial and exponential. A second reason comes from Theorem 6.1.1: it is natural to compare the homological sequences of N over φ to

$(\beta_n^R(k))_{n\geqslant 0}$, and this sequence is known to have either polynomial or exponential growth.

7.1. Complexities and curvatures. The *complexity* of N over φ is the number

$$\operatorname{cx}_\varphi N = \inf\left\{ d\in\mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a number } c\in\mathbb{R} \text{ such that} \\ \beta_n^\varphi(N)\le cn^{d-1} \text{ for all } n\gg 0 \end{array}\right\}.$$

The *curvature* of N over φ is the number

$$\operatorname{curv}_\varphi N = \limsup_n \sqrt[n]{\beta_n^\varphi(N)}.$$

Similar formulas, in which Betti numbers are replaced by Bass numbers, define the *injective complexity* $\operatorname{inj\,cx}_\varphi N$ and the *injective curvature* $\operatorname{inj\,curv}_\varphi N$ of N over φ.

When $\varphi = \mathrm{id}^R$ one speaks of complexities and curvatures of N over R, modifying the notation accordingly to $\operatorname{cx}_R(N)$, etc., see [5]. From Proposition 4.3.2 one gets:

Remark 7.1.1. If $R \xrightarrow{\dot\varphi} R' \xrightarrow{\varphi'} \widehat{S}$ is any Cohen factorization of φ', then

$$\operatorname{cx}_\varphi N = \operatorname{cx}_{\dot\varphi}\widehat{N} = \operatorname{cx}_{\widehat{\varphi}}\widehat{N} = \operatorname{cx}_{R'}(\widehat{N})$$
$$\operatorname{curv}_\varphi N = \operatorname{curv}_{\dot\varphi}\widehat{N} = \operatorname{curv}_{\widehat{\varphi}}\widehat{N} = \operatorname{curv}_{R'}(\widehat{N}).$$

The corresponding injective invariants satisfy analogous relations.

The equalities below often allow one to restrict proofs to projective invariants.

Remark 7.1.2. If D is a dualizing complex of $\widehat{S}$ and $\widehat{N}^\dagger = \mathbf{R}\mathrm{Hom}_{\widehat{S}}\left(\widehat{N}, D\right)$, then

$$\operatorname{inj\,cx}_\varphi N = \operatorname{cx}_{\dot\varphi}\widehat{N}^\dagger \qquad \text{and} \qquad \operatorname{inj\,curv}_\varphi N = \operatorname{curv}_{\dot\varphi}\widehat{N}^\dagger.$$

Indeed, this is an immediate consequence of Theorem 4.2.2.

We list some basic dependencies among homological invariants of N.

PROPOSITION 7.1.3. *The numbers* $\operatorname{cx}_\varphi N$ *and* $\operatorname{curv}_\varphi N$ *satisfy the relations below.*
(1) $\operatorname{flat\,dim}_R N < \infty$ *if and only if* $\operatorname{cx}_\varphi N = 0$, *if and only if* $\operatorname{curv}_\varphi N = 0$.
(2) *If* $\operatorname{flat\,dim}_R N = \infty$, *then* $\operatorname{cx}_\varphi N \ge 1$ *and* $\operatorname{curv}_\varphi N \ge 1$.
(3) *If* $1 \le \operatorname{cx}_\varphi N < \infty$, *then* $\operatorname{curv}_\varphi N = 1$.
(4) *If* $\operatorname{curv}_\varphi N > 1$, *then there exist an infinite sequence* $n_1 < n_2 < \dots$ *of integers and a real number* $b > 1$, *such that* $\beta_{n_i}^\varphi(N) \ge b^{n_i}$ *holds for all* $i \ge 1$.

(5) $\operatorname{curv}_\varphi N \leq \operatorname{curv}_R k < \infty$.
The corresponding injective invariants satisfy analogous relations.

Remark. Part (5) shows, in particular, that there exists a positive real number c, such that $\beta^\varphi_n(N) \leq c^n$ and $\mu^n_\varphi(N) \leq c^n$ hold for all $n \geq 1$.

Proof. By Remark 7.1.2 we may restrict to projective invariants. Part (1) comes from Remark 4.1.4 and Corollary 2.5. Parts (2), (3), and (4) are clear. In (5) the first inequality comes from Theorem 6.1.1, see (7.1.6) below. The second inequality is known, see e.g. [6, (4.2.3)]; a self-contained proof is given in Corollary 9.2.2. □

Complexity may be infinite, even when φ is the identity map.

Example 7.1.4. For the local ring $R = k[x, y]/(x^2, xy, y^2)$ and the R-module $N = R/(x, y)$ one has $\beta^R_n(N) = 2^n$, so $\operatorname{cx}_R N = \infty$.

We determine when *all* homologically finite complexes have finite complexity. Recall that $\mathcal{D}^{\mathrm{f}}(S)$ denotes the derived category of homologically finite complexes.

THEOREM 7.1.5. *Set* $c = \operatorname{codim} R$. *The following conditions are equivalent.*
(i) *R is a local complete intersection ring.*
(ii) $\operatorname{cx}_\varphi X \leq c$ *for each* $X \in \mathcal{D}^{\mathrm{f}}(S)$.
(iii) $\operatorname{cx}_\varphi l = c$.
(iv) $\operatorname{curv}_\varphi l \leq 1$.
They are also equivalent to those obtained by replacing Poincaré series, complexity, curvature with Bass series, injective complexity, injective curvature, respectively.

Proof. As l is separated over φ by Example 6.1.3, from (7.1.6) below one gets

$$\operatorname{cx}_\varphi l = \operatorname{cx}_R k \qquad \text{and} \qquad \operatorname{curv}_\varphi l = \operatorname{curv}_R k.$$

(i) ⇒ (iii). As $P^R_k(t) = (1 + t)^{\operatorname{edim} R}/(1 - t^2)^c$, see (6.3.6.2), one has $\operatorname{cx}_R k = c$ by (7.1.6.4), so the equalities above yield $\operatorname{cx}_\varphi l = c$.
(iii) ⇒ (ii) holds by Theorem 6.1.1.
(ii) ⇒ (iii) ⇒ (iv) hold by definition.
(iv) ⇒ (i) is a consequence of the formulas above and (6.3.6.3). □

Our results on complexities and curvatures are often deduced from coefficientwise inequalities of formal Laurent series. We collect some simple rules of operation.

7.1.6. Let $a(t) = \sum a_n t^n$ be a formal Laurent series with nonnegative integer coefficients. Extending notation, let $\operatorname{cx}(a(t))$ denote the least integer d such that $a_n \leq cn^{d-1}$ holds for some $c \in \mathbb{R}$ and all $n \gg 0$, and set $\operatorname{curv}(a(t)) = \limsup_n \sqrt[n]{a_n}$.

Let $b(t)$ be a formal Laurent series with nonnegative integer coefficients.

(1) If $a(t) \preccurlyeq b(t)$, then

$$\operatorname{cx}(a(t)) \leq \operatorname{cx}(b(t)) \quad \text{and} \quad \operatorname{curv}(a(t)) \leq \operatorname{curv}(b(t)).$$

(2) The following inequalities hold:

$$\begin{aligned} \min\{\operatorname{cx}(a(t)),\, \operatorname{cx}(b(t))\} &\leq \operatorname{cx}(a(t)\cdot b(t)) \leq \operatorname{cx}(a(t)) + \operatorname{cx}(b(t)) \\ \min\{\operatorname{curv}(a(t)),\, \operatorname{curv}(b(t))\} &\leq \operatorname{curv}(a(t)\cdot b(t)) \\ &\leq \max\{\operatorname{curv}(a(t)),\, \operatorname{curv}(b(t))\}. \end{aligned}$$

When $a(t)$ represents a rational function the following hold as well.

(3) $\operatorname{curv}(a(t))$ is finite.

(4) $\operatorname{cx}(a(t))$ is finite if and only if $a(t)$ converges in the unit circle; when this is the case $\operatorname{cx}(a(t))$ is equal to the order of the pole of $a(t)$ at $t = 1$;

(5) If $b(t)$ represents a rational function, and $\operatorname{cx}(a(t))$, $\operatorname{cx}(b(t))$ are finite, then

$$\operatorname{cx}(a(t)\cdot b(t)) = \operatorname{cx}(a(t)) + \operatorname{cx}(b(t)).$$

Indeed, (1) and the first three inequalities in (2) follow from the definitions. Since $\operatorname{curv}(a(t))$ is the inverse of the radius of convergence of the series $a(t)$, the last inequality in (2) is a reformulation of a well known property of power series, and (3) is clear. For (4), see e.g. [4, (2.4)]; (5) follows from (4).

7.2. Reductions. The determination of complexity or curvature can sometimes be simplified using Koszul complexes. The following result is a step in that direction.

Proposition 7.2.1. *If* $\mathbf{v}$ *is a finite subset of* $\mathfrak{n}$*, then*

$$\operatorname{cx}_\varphi N = \operatorname{cx}_\varphi(\mathrm{K}[\mathbf{v}; N]) \qquad \textit{and} \qquad \operatorname{curv}_\varphi N = \operatorname{curv}_\varphi(\mathrm{K}[\mathbf{v}; N]).$$

Similar equalities hold for the corresponding injective invariants.

Proof. In view of the isomorphism of complexes $\mathrm{K}[\mathbf{x}; \mathrm{K}[\mathbf{v}; N]] \cong \mathrm{K}[\mathbf{x}, \mathbf{v}; N]$, Proposition 4.3.1 applied to the set $\mathbf{y} = \mathbf{x} \sqcup \mathbf{v}$ yields

$$P^{\varphi}_{\mathrm{K}[\mathbf{v};N]}(t) = P^{\varphi}_N(t)\cdot(1+t)^{\operatorname{card}\mathbf{v}}.$$

The equalities for complexity and curvature now result from (7.1.6), and those for their injective counterparts follow via (3.6) and Theorem 4.2.2. □

COROLLARY 7.2.2. *If L is an S-module and $\mathbf{v}$ is an L-regular subset of S, then*

$$\operatorname{cx}_\varphi(L/\mathbf{v}L) = \operatorname{cx}_\varphi L \qquad \textit{and} \qquad \operatorname{curv}_\varphi(L/\mathbf{v}L) = \operatorname{curv}_\varphi L.$$

Similar equalities hold for the corresponding injective invariants.

Proof. In this case $L/\mathbf{v}L$ is isomorphic to $\mathrm{K}[\mathbf{v};L]$ in the derived category of S-modules. □

As a special case of the next theorem, the Koszul complex on any system of parameters for the S-module $\mathrm{H}(N)/\mathfrak{m}\mathrm{H}(N)$ can be used to determine the complexity or the curvature of N over φ.

THEOREM 7.2.3. *Let $\mathbf{v}$ be a finite subset of $\mathfrak{n}$. If the S-module*

$$\frac{\mathrm{H}(N)}{\mathbf{v}\mathrm{H}(N)+\mathfrak{m}\mathrm{H}(N)}$$

has finite length, then

$$\operatorname{cx}_\varphi N = \operatorname{cx}\left(\sum_{n\in\mathbb{Z}} \ell_S \operatorname{Tor}^R_n(k, \mathrm{K}[\mathbf{v};N])\, t^n\right)$$

$$\operatorname{curv}_\varphi N = \limsup_n \sqrt[n]{\ell_S \operatorname{Tor}^R_n(k, \mathrm{K}[\mathbf{v};N])}.$$

Similar equalities involving $\ell_S \operatorname{Ext}^n_R(k, \mathrm{K}[\mathbf{v};N])$ express $\operatorname{inj\,cx}_\varphi N$ *and* $\operatorname{inj\,curv}_\varphi N$.

Remark 7.2.4. When $\ell_S\left(\mathrm{H}(N)/\mathfrak{m}\mathrm{H}(N)\right)$ is finite one may choose $\mathbf{v} = \varnothing$, so that $\mathrm{K}[\mathbf{v};N] = N$; this applies, in particular, when the ring $S/\mathfrak{m}S$ is artinian.

Proof of Theorem 7.2.3. The proofs of the formulas for complexity and curvature are similar to those for their injective counterparts; for once, we verify the latter.

Set $K = \mathrm{K}[\mathbf{v};N]$. By Proposition 7.2.1, it suffices to prove that the expressions on the right hand sides of the desired equalities are equal to $\operatorname{inj\,cx}_\varphi K$ and $\operatorname{inj\,curv}_\varphi K$, respectively. Note the equalities of supports of S-modules

$$\begin{aligned}
\operatorname{Supp}_S\left(\frac{\mathrm{H}(N)}{\mathbf{v}\mathrm{H}(N)+\mathfrak{m}\mathrm{H}(N)}\right) &= \operatorname{Supp}_S(S/\mathfrak{m}S)\cap\operatorname{Supp}_S(S/\mathbf{v}S)\cap\operatorname{Supp}_S\mathrm{H}(N)\\
&= \operatorname{Supp}_S(S/\mathfrak{m}S)\cap\operatorname{Supp}_S\mathrm{H}(K)\\
&= \operatorname{Supp}_S\left(\mathrm{H}(K)/\mathfrak{m}\mathrm{H}(K)\right),
\end{aligned}$$

where the middle one comes from Lemma 1.5.5 and the other two are standard. Therefore, the length of the S-module $\mathrm{H}(K)/\mathfrak{m}\mathrm{H}(K)$ is finite.

We claim that there exists a positive integer v for which there are inequalities:

$$I_{\varphi}^{K}(t)t^{\operatorname{edim}\varphi} \preccurlyeq \sum_{n\in\mathbb{Z}} \ell_S\left(\operatorname{Ext}_R^n(k,K)\right) t^n \cdot (1+t)^{\operatorname{edim}\varphi} \preccurlyeq I_{\varphi}^{K}(t)\cdot(vt)^{\operatorname{edim}\varphi}.$$

Indeed, let $\mathbf{x} = \{x_1,\dots,x_e\}$ be a minimal generating set of $\mathfrak{n}$ modulo $\mathfrak{m}S$. Set $K^{(j)} = \mathrm{K}[x_1,\dots,x_j;K]$ for $j = 0,\dots,e$, and note that $K^{(j+1)}$ is the mapping cone of the morphism $\lambda^{(j)}\colon K^{(j)} \to K^{(j)}$ given by multiplication with x_{j+1}, cf. (1.5.4).

By (1.4.3), our hypothesis implies that there is an integer $v \geq 1$ such that $\mathfrak{n}^v$ is contained in $\mathfrak{m}S+\operatorname{Ann}_{\mathcal{D}(S)}(K)$. This ideal is contained in $\mathfrak{m}S+\operatorname{Ann}_{\mathcal{D}(S)}(K^{(j)})$ for each j, so by Lemma 1.5.6 each S-module $\mathrm{H}(k\otimes_R^{\mathbf{L}} K^{(j)})$ is finite and annihilated by $\mathfrak{n}^v$. Furthermore, there is a triangle

$$k\otimes_R^{\mathbf{L}} K^{(j)} \xrightarrow{k\otimes_R^{\mathbf{L}}\lambda^{(j)}} k\otimes_R^{\mathbf{L}} K^{(j)} \longrightarrow k\otimes_R^{\mathbf{L}} K^{(j+1)} \longrightarrow .$$

In view of Lemma 1.2.3, the formal Laurent series

$$I^{(j)}(t) = \sum_{n\in\mathbb{Z}} \ell_S(\operatorname{Ext}_R^n(k,K^{(j)}))\, t^n$$

satisfy coefficientwise inequalities

$$I^{(j+1)}(t) \preccurlyeq I^{(j)}(t)\cdot(1+t) \preccurlyeq I^{(j+1)}(t)\cdot v.$$

It remains to note that $I^{(e)}(t) = I_{\varphi}^{K}(t)\cdot t^{\operatorname{edim}\varphi}$. □

8. Comparisons. The source ring R and the target ring S of the homomorphism φ play different roles in the definitions of Betti numbers and Bass numbers, and hence in the characteristics of N introduced above: The impact of the R-module structure of N is through the entire resolution of k, while that of its S-module structure is limited to the Koszul complex $\mathrm{K}^S[N]$. We take a closer look at the role of S.

For the rest of the section we fix a second local homomorphism

$$\widetilde{\varphi}\colon (R,\mathfrak{m},k) \to (\widetilde{S},\widetilde{\mathfrak{n}},\widetilde{l})$$

and we assume that N also has a structure of a homologically finite complex of $\widetilde{S}$-modules, and that the action of R through $\widetilde{\varphi}$ coincides with that through φ.

Our main conclusion is that if the actions of S and $\widetilde{S}$ on N commute, then the (injective) complexity or curvature of N over φ is equal to that over $\widetilde{\varphi}$. This may be surprising, as it is in general impossible to express the Betti numbers or the Bass numbers over one of the maps in terms of those over the other. As an application, we show that in the presence of commuting actions important invariants of N over S, such as depth and Krull dimension, are equal to those over $\widetilde{S}$.

8.1. Asymptotic invariants. We consider the following natural questions.

Question 8.1.1. Do the equalities below always hold:

$$\operatorname{cx}_{\varphi} N = \operatorname{cx}_{\widetilde{\varphi}} N \qquad \text{and} \qquad \operatorname{curv}_{\varphi} N = \operatorname{curv}_{\widetilde{\varphi}} N?$$

Question 8.1.2. What about the corresponding injective invariants?

The next result settles an important special case. While we do not know the answer in general, Example 8.2.3 raises the possibility of a negative answer.

THEOREM 8.1.3. *If the actions of S and $\widetilde{S}$ on N commute, then*

$$\operatorname{cx}_{\varphi} N = \operatorname{cx}_{\widetilde{\varphi}} N \qquad \textit{and} \qquad \operatorname{curv}_{\varphi} N = \operatorname{curv}_{\widetilde{\varphi}} N$$

Similar equalities also hold for the injective invariants of N.

The special case where $\varphi' = \mathrm{id}^R$ is of independent interest.

COROLLARY 8.1.4. *If* $\mathrm{H}(N)$ *is finite over R, then*

$$\operatorname{cx}_{\varphi} N = \operatorname{cx}_{R} N \qquad \textit{and} \qquad \operatorname{curv}_{\varphi} N = \operatorname{curv}_{R} N$$

Similar equalities for $\operatorname{inj\,cx} N$ *and* $\operatorname{inj\,curv} N$ *also hold.*

The theorem is a consequence of the relations of formal Laurent series established in the proposition below. Indeed, (7.1.6) yields $\operatorname{cx}_{\varphi} N \leq \operatorname{cx}_{\widetilde{\varphi}} N$ and $\operatorname{curv}_{\varphi} N \leq \operatorname{curv}_{\widetilde{\varphi}} N$, and the converse inequalities follow by symmetry.

PROPOSITION 8.1.5. *If the actions of S and $\widetilde{S}$ commute, then there exists a positive integer w, such that*

$$P_N^{\varphi}(t) \cdot (1+t)^{\operatorname{edim} \widetilde{\varphi}} \preccurlyeq P_N^{\widetilde{\varphi}}(t) \cdot w\,(1+t)^{\operatorname{edim} \varphi}$$
$$I_{\varphi}^N(t) \cdot (1+t)^{\operatorname{edim} \widetilde{\varphi}} \preccurlyeq I_{\widetilde{\varphi}}^N(t) \cdot w\,(1+t)^{\operatorname{edim} \varphi}.$$

The proposition is proved at the end of the section. Next we use it to compare patterns of vanishing of Betti numbers and Bass numbers over φ and over $\widetilde{\varphi}$.

8.2. Homological dimensions. We define the *projective dimension* of N over φ and the *injective dimension* of N over φ, respectively, to be the numbers

$$\begin{aligned}\operatorname{proj\,dim}_\varphi N &= \sup\{n\in\mathbb{Z} \mid \beta_n^\varphi(N)\neq 0\} - \operatorname{edim}\varphi\\ \operatorname{inj\,dim}_\varphi N &= \sup\{n\in\mathbb{Z} \mid \mu_\varphi^n(N)\neq 0\} - \operatorname{edim}\varphi.\end{aligned}$$

These expressions have been chosen in view of the characterizations of homological dimensions over R recalled in (2.2). The shifts by $\operatorname{edim}\varphi$ appear because the modified invariants appear to carry a more transparent algebraic meaning.

The equalities below come from Remark 4.1.4, and reconcile our notion of projective dimension over φ with that defined in [20, (3.5)] via Cohen factorizations.

8.2.1. Let $\grave{\varphi}\colon R\to\widehat{S}$ be the composition of φ with the completion $S\to\widehat{S}$. If $R\to R'\to\widehat{S}$ is a minimal Cohen factorization of $\grave{\varphi}$, then

$$\begin{aligned}\operatorname{proj\,dim}_\varphi N = \operatorname{proj\,dim}_{\grave{\varphi}}\widehat{N} = \operatorname{proj\,dim}_{\widehat{\varphi}}\widehat{N} &= \operatorname{flat\,dim}_{R'}\widehat{N} - \operatorname{edim}\varphi\\ \operatorname{inj\,dim}_\varphi N = \operatorname{inj\,dim}_{\grave{\varphi}}\widehat{N} = \operatorname{inj\,dim}_{\widehat{\varphi}}\widehat{N} &= \operatorname{inj\,dim}_{R'}\widehat{N} - \operatorname{edim}\varphi.\end{aligned}$$

Combining these equalities with Corollary 2.5, we obtain:

COROLLARY 8.2.2. *The following (in)equalities hold:*

$$\begin{gathered}\operatorname{flat\,dim}_R N - \operatorname{edim}\varphi \le \operatorname{proj\,dim}_\varphi N \le \operatorname{flat\,dim}_R N\\ \operatorname{inj\,dim}_\varphi N = \operatorname{inj\,dim}_R N.\end{gathered}\qquad\square$$

Corollary 8.2.2, applied to φ and $\widetilde{\varphi}$, implies that N has finite projective dimension over φ if and only if it does over $\widetilde{\varphi}$. However, unlike injective dimension, the *value* of $\operatorname{proj\,dim}_\varphi N$ may depend on the map.

Example 8.2.3. Let R be a field. Set $S = R[x]_{(x)}$ and let $\varphi\colon R\to S$ be the canonical injection. Set $\widetilde{S} = R(x)$ and let $\widetilde{\varphi}\colon R\to\widetilde{S}$ be the canonical injection. Decomposition into partial fractions shows that S and $\widetilde{S}$ have the same rank as vector spaces over R, namely, $c = \max\{\aleph_0, \operatorname{card}(R)\}$. Pick an R-vector space N of rank c. Choosing R-linear isomorphisms $S\cong N$ and $\widetilde{S}\cong N$, endow N with structures of free module of rank 1 over S and over $\widetilde{S}$; both actions restrict to the original action of R on N. Directly, or from (8.2.1), one gets

$$\operatorname{proj\,dim}_\varphi N = -\operatorname{edim} S = -1 \neq 0 = -\operatorname{edim}\widetilde{S} = \operatorname{proj\,dim}_{\widetilde{\varphi}} N.$$

The module in the example is *not* an S-$\widetilde{S}$-bimodule. There is a reason:

THEOREM 8.2.4. *If the actions of S and $\widetilde{S}$ on N commute, then*

$$\operatorname{proj\,dim}_{\varphi} N = \operatorname{proj\,dim}_{\widetilde{\varphi}} N.$$

Proof. Comparing the degrees of the Laurent series in Proposition 8.1.5, one gets

$$(\operatorname{proj\,dim}_{\varphi} N + \operatorname{edim}\varphi) + \operatorname{edim}\widetilde{\varphi} \leq (\operatorname{proj\,dim}_{\widetilde{\varphi}} N + \operatorname{edim}\widetilde{\varphi}) + \operatorname{edim}\varphi.$$

Thus, $\operatorname{proj\,dim}_{\varphi} N \leq \operatorname{proj\,dim}_{\widetilde{\varphi}} N$; the converse inequality holds by symmetry. □

8.3. Depth and Krull dimension. The definition of depth of N over S is recalled in (3.3). When $\operatorname{proj\,dim}_{\varphi} N$ is finite, it appears in the following equality of Auslander-Buchsbaum type, proved in [20, (4.3)]:

$$\operatorname{depth}_S N = \operatorname{depth} R - \operatorname{proj\,dim}_{\varphi} N.$$

Foxby [16, (3.5)] defines the (*Krull*) *dimension* of N over S to be

$$\dim_S N = \sup\{\dim_S \mathrm{H}_n(N) - n \mid n \in \mathbb{Z}\}.$$

Clearly, this number specializes to the usual concept when N is an S-module.

The module N in Example 8.2.3 has $\dim_S N = \operatorname{depth}_S N = 1$ and $\dim_{\widetilde{S}} N = \operatorname{depth}_{\widetilde{S}} N = 0$. This explains the interest of the following theorem. Unlike most results in this paper, it is *not* about invariants over a homomorphism; in particular, we are not assuming that Q is an R-algebra.

THEOREM 8.3.1. *Let Q be a local ring such that N has a structure of homologically finite complex of Q-modules. If the actions of Q and S on N commute, then*

$$\dim_Q N = \dim_S N \qquad \textit{and} \qquad \operatorname{depth}_Q N = \operatorname{depth}_S N.$$

Proof. We may assume $\mathrm{H}_n(N) = H \neq 0$ for some $n \in \mathbb{Z}$. Let q be the characteristic of the residue field of Q, and p be that of l. We claim $q = p$. There is nothing to prove unless p or q is positive, and then we may assume $q > 0$. As $V = H/\mathfrak{n}H$ is a finite Q-module, and q is contained in the maximal ideal of Q, Nakayama's Lemma yields $qV \neq V$. Since V is a l-vector space, this implies $qV = 0$, hence $q = p$.

Set $P = \mathbb{Z}_{(p)}$ and let $\eta\colon\ P \to Q$ and $\eta'\colon\ P \to S$ be the canonical maps.

First, we verify the assertion on depths: since the actions of Q and S on N commute, $\operatorname{proj\,dim}_{\eta} N = \operatorname{proj\,dim}_{\eta'} N$ by Theorem 8.2.4. The local ring P is regular, so $\operatorname{flat\,dim}_P N$ is finite, and hence, by (8.2.2), both $\operatorname{proj\,dim}_{\eta} N$ and $\operatorname{proj\,dim}_{\eta'} N$ are finite. The Auslander-Buchsbaum formula, see above, yields

$$\operatorname{depth}_Q N = \operatorname{depth} R - \operatorname{proj\,dim}_{\eta} N = \operatorname{depth} R - \operatorname{proj\,dim}_{\eta'} N = \operatorname{depth}_S N.$$

Now we turn to dimensions. It suffices to check that $\dim_Q \mathrm{H}_n(N) = \dim_S \mathrm{H}_n(N)$ for each $n \in \mathbb{Z}$, so we may assume that N is a module, concentrated in degree zero. The actions of Q and S on N commute, so one obtains a homomorphism of rings

$$\tau\colon\ Q \otimes_P S \longrightarrow \operatorname{Hom}_P(N, N) \quad \text{where} \quad (q \otimes s) \longmapsto (n \mapsto qsn)\,.$$

Set $U = \operatorname{Im}(\tau)$: This is a commutative subring of $\operatorname{Hom}_P(N, N)$, so N has a natural U-module structure. The actions of U and Q commute, so we have inclusions

$$U \subseteq \operatorname{Hom}_Q(N, N) \subseteq \operatorname{Hom}_P(N, N)$$

where the first one is Q-linear. Since N is a finite Q-module, the same is true of $\operatorname{Hom}_Q(N, N)$, and hence of U. Therefore, the composed homomorphism of rings $Q \to Q \otimes_P S \to U$ is module finite. Thus, U is noetherian and $\dim_Q N = \dim_U N$. By symmetry, $\dim_S N = \dim_U N$, hence $\dim_Q N = \dim_S N$, as desired. $\square$

The preceding theorem allows us to complement a result in [20]. In that paper a notion of Gorenstein dimension of N over φ, denoted $\mathrm{G}-\dim_{\varphi} N$, is defined and studied. In particular, it is proved in [20, (7.1)] that $\mathrm{G}-\dim_{\varphi} N$ and $\mathrm{G}-\dim_{\widetilde{\varphi}} N$ are simultaneously finite. However, they can differ, as shown by an example in [20, (7.2)]; there, as in Example 8.2.3, the actions of S and $\widetilde{S}$ do not commute.

Remark 8.3.2. If the actions of S and $\widetilde{S}$ on N commute, then

$$\mathrm{G}-\dim_{\varphi} N = \mathrm{G}-\dim_{\widetilde{\varphi}} N.$$

Indeed, it is clear from the preceding discussion that we may assume both Gorenstein dimensions are finite. In that case, we have

$$\mathrm{G}-\dim_{\varphi} N = \operatorname{depth} R - \operatorname{depth}_S N = \operatorname{depth} R - \operatorname{depth}_{\widetilde{S}} N = \mathrm{G}-\dim_{\widetilde{\varphi}} N$$

where the equalities on both ends are given by [20, (3.5)], and the one in the middle comes from Theorem 8.3.1.

8.4. Poincaré series. Now we turn to the proof of Proposition 8.1.5. The argument hinges on the following construction.

Construction 8.4.1. Each complex Y of modules over the ring $T = S \otimes_R \widetilde{S}$ defines a commutative diagram of homomorphisms of rings

$$\begin{tikzcd}
\operatorname{Ext}^0_T(Y,Y) \arrow[rrr] \arrow[ddd] & & & \operatorname{Ext}^0_S(Y,Y) \arrow[ddd] \\
& T \arrow[ul, "\eta_T"] & S \arrow[l] \arrow[ur, "\eta_S"'] & \\
& \widetilde{S} \arrow[u] \arrow[dl, "\eta_{\widetilde{S}}"'] & R \arrow[l, "\widetilde{\varphi}"] \arrow[u, "\varphi"'] \arrow[dr, "\eta_R"] & \\
\operatorname{Ext}^0_{\widetilde{S}}(Y,Y) \arrow[rrr] & & & \operatorname{Ext}^0_R(Y,Y)
\end{tikzcd}$$

where the slanted arrows land into central subrings.

It is assembled is follows. The inner square is the canonical commutative diagram associated with a tensor product of R-algebras. The outer square is obtained from it by functoriality, hence it commutes. The slanted arrows refer to the maps described in (1.4), so they are central and the trapezoids commute.

Let $\tau\colon\ T \to \operatorname{Ext}^0_R(Y,Y)$ be the homomorphism of provided by the diagram above. Setting $U = \operatorname{Im}(\tau)$, one obtains a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\varphi"'] & \widetilde{S} \arrow[d, "\sigma'"] \\
S \arrow[r, "\sigma"'] & U
\end{tikzcd}$$

of homomorphisms of commutative rings.

LEMMA 8.4.2. *With the notation of Construction 8.4.1, let $\mathfrak{r}$ denote the Jacobson radical of the ring U, set $V_n = \operatorname{Tor}^R_n(k,Y)$, respectively, $V_n = \operatorname{Ext}^{-n}_R(k,Y)$ for all $n \in \mathbb{Z}$, and assume that Y is homologically finite over S and over $\widetilde{S}$.*

(1) *The ring U is finite as an S-module,* $\mathfrak{r} = \operatorname{rad}(\mathfrak{n}U)$*, and*

$$\inf\{j \in \mathbb{N} \mid \mathfrak{r}^j \subseteq \mathfrak{n}U\} = v < \infty.$$

(2) *For $u = \ell_S(U/\mathfrak{r})$ the following inequalities hold:*

$$\ell_T(V_n) \leq \ell_S(V_n) \leq u\ell_T(V_n).$$

(3) *If $\ell_S(V_j)$ is finite for all $j \in \mathbb{Z}$, then for each $\widetilde{x} \in \widetilde{\mathfrak{n}}$ the S-modules $\widetilde{V}_n = \operatorname{Tor}_n^R(k, \mathrm{K}[\widetilde{x}; Y])$, respectively, $\widetilde{V}_n = \operatorname{Ext}_R^{-n}(k, \mathrm{K}[\widetilde{x}; Y])$, satisfy*

$$\frac{1}{v}\big(\ell_S(V_n) + \ell_S(V_{n-1})\big) \leq \ell_S(\widetilde{V}_n) \leq \ell_S(V_n) + \ell_S(V_{n-1}).$$

Proof. (1) The commutative diagram in Construction 8.4.1 shows that the maps

$$T \to \operatorname{Ext}_S^0(Y, Y) \to \operatorname{Ext}_R^0(Y, Y)$$

are S-linear. Thus, U is an S-submodule of the image of $\operatorname{Ext}_S^0(Y, Y)$. The last S-module is finite by (1.3.2), hence so is U. The Going-up Theorem now implies $\mathfrak{r} = \operatorname{rad}(\mathfrak{n}U)$. Since $U/\mathfrak{n}U$ is a finite l-algebra, its Jacobson radical $\mathfrak{r}/\mathfrak{n}U$ is nilpotent, hence $\mathfrak{r}^v \subseteq \mathfrak{n}$ for some v.

(2) It follows from (1.3.1) that T acts on V_n through the map τ. Since for every U-module V one has $\ell_T(V) = \ell_U(V)$, it suffices to prove the inequalities

$$\ell_S(U/\mathfrak{r}) < \infty \qquad \text{and} \qquad \ell_U(V) \leq \ell_S(V) \leq \ell_U(V) \cdot \ell_S(U/\mathfrak{r}).$$

The first one is an immediate consequence of (1). When V is simple $\mathfrak{r}V = 0$ and $\ell_U(V) = 1$, so the last two inequalities are clear in this case. The general case follows by computing lengths of subquotients of the filtration $\{\mathfrak{r}^i V\}_{i \geq 0}$.

(3) For this argument we identify $\mathrm{K}[\widetilde{x}; Y]$ with the mapping cone of $\lambda_{\widetilde{x}}^Y$, see (1.5.4), and set $W = k \otimes_R^{\mathbf{L}} Y$. This leads to an equality of morphisms

$$k \otimes_R^{\mathbf{L}} \lambda_{\widetilde{x}}^Y = \lambda_{\widetilde{x}}^W \colon W \longrightarrow W$$

and to an identification of the complex $k \otimes_R^{\mathbf{L}} \mathrm{K}[\widetilde{x}; Y]$ with the mapping cone of $\lambda_{\widetilde{x}}^W$. For $y = \sigma'(\widetilde{x}) \in U$ we now obtain

$$\mathrm{H}(\lambda_{\widetilde{x}}^W) = \lambda_{\widetilde{x}}^{\mathrm{H}(W)} = \lambda_y^{\mathrm{H}(W)}.$$

Applied first with $\widetilde{S}$ in place of S, Part (1) yields $\mathfrak{r} = \operatorname{rad}(\widetilde{\mathfrak{n}}U)$, hence $y \in \mathfrak{r}$; applied then to S, it gives $y^v \in \mathfrak{n}U$. By (1.5.6), the ideal $\mathfrak{n}$ annihilates V_n, so the equalities above yield $\mathrm{H}(\lambda_{\widetilde{x}}^W)^v = 0$. The desired result now follows from Lemma 1.2.3. □

Proof of Theorem 8.1.5. We only give the proof for Poincaré series.

Let $\mathbf{x} = \{x_1, \dots, x_e\}$ and $\widetilde{\mathbf{x}} = \{\widetilde{x}_1, \dots, \widetilde{x}_g\}$ be minimal sets of generators of $\mathfrak{n}$ modulo $\mathfrak{m}S$ and $\widetilde{\mathfrak{n}}$ modulo $\mathfrak{m}\widetilde{S}$, respectively. For any finite set $\mathbf{y}$ in T, set

$$F^S_{\mathbf{y}}(t) = \sum_{n\in\mathbb{Z}} \ell_S\left(\operatorname{Tor}^R_n(k, \mathrm{K}[\mathbf{y}; N])\right) t^n.$$

By definition, one has $F^S_{\mathbf{x}}(t) = P^{\varphi}_N(t)$ and $F^{\widetilde{S}}_{\widetilde{\mathbf{x}}}(t) = P^{\widetilde{\varphi}}_N(t)$, so the desired formula follows from the chain of inequalites

$$\begin{aligned}
F^S_{\mathbf{x}}(t)\cdot(1+t)^g &\preccurlyeq F^S_{\mathbf{x}\sqcup\{\widetilde{x}_1\}}(t)\cdot(1+t)^{g-1}\,v_1 \preccurlyeq \cdots \\
&\preccurlyeq F^S_{\mathbf{x}\sqcup\widetilde{\mathbf{x}}}(t)\cdot v_1\cdots v_g \\
&\preccurlyeq F^T_{\mathbf{x}\sqcup\widetilde{\mathbf{x}}}(t)\cdot v_1\cdots v_g\cdot u \\
&\preccurlyeq F^{\widetilde{S}}_{\mathbf{x}\sqcup\widetilde{\mathbf{x}}}(t)\cdot v_1\cdots v_g\cdot u \\
&\preccurlyeq F^{\widetilde{S}}_{\mathbf{x}\smallsetminus\{x_e\}\sqcup\widetilde{\mathbf{x}}}(t)\cdot(1+t)\,v_1\cdots v_g\cdot u \preccurlyeq \cdots \\
&\preccurlyeq F^{\widetilde{S}}_{\widetilde{\mathbf{x}}}(t)\cdot(1+t)^e v_1\cdots v_g\cdot u
\end{aligned}$$

obtained as follows. The inequalities in the first two rows come from the left-hand side of the sandwich in Lemma 8.4.2.3, applied successively to the complexes of S-modules $\mathrm{K}[\mathbf{x}\sqcup\{\widetilde{x}_1,\dots,\widetilde{x}_j\}; N]$ for $j = 0,\dots,g-1$. The next pair of inequalities are provided by Lemma 8.4.2.2, applied to the complex $\mathrm{K}[\mathbf{x}\sqcup\widetilde{\mathbf{x}}; N]$ considered first over S, then over $\widetilde{S}$. The final string of e inequalities is obtained from the right hand side of the sandwich in Lemma 8.4.2.3, applied successively to the complexes of $\widetilde{S}$-modules $\mathrm{K}[\{x_1,\dots,x_{e-j}\}\sqcup\widetilde{\mathbf{x}}; N]$ for $j = 1,\dots,e$. □

9. Composition. In this section we study how the sequences of Betti numbers and Bass numbers over φ react to various changes of rings. Most of the results take the form of coefficientwise equalities and inequalities involving their generating functions. This format is well adopted to study the finiteness and the asymptotic behavior of the Betti numbers and the Bass numbers. In particular, it does not depend on the choice of scale used to measure asymptotic growth: this fact might acquire importance should future investigations discover modules or complexes whose Betti numbers have superpolynomial, but subexponential rates of growth.

Throughout the section we fix a local homomorphism

$$\psi\colon (Q,\mathfrak{l},h)\to(R,\mathfrak{m},k).$$

9.1. Upper bounds for compositions. We bound (injective) complexities and curvatures over $\varphi\circ\psi$ in terms of invariants over the maps ψ and φ.

THEOREM 9.1.1. *The following relations hold:*
(1) *The following inequalities hold.*

$$\begin{aligned}\operatorname{cx}_{\varphi\circ\psi} N &\le \operatorname{cx}_{\varphi} N + \operatorname{cx}_{\psi} R\\ \operatorname{curv}_{\varphi\circ\psi} N &\le \max\{\operatorname{curv}_{\varphi} N, \operatorname{curv}_{\psi} R\}.\end{aligned}$$

(2) *If* $\mathrm{H}(N)$ *is finite over* R*, then*

$$\operatorname{cx}_{\varphi\circ\psi} N = \operatorname{cx}_{\psi} N \quad \textit{and} \quad \operatorname{curv}_{\varphi\circ\psi} N = \operatorname{curv}_{\psi} N.$$

In addition, the relations obtained by replacing complexities and curvatures of N *with their injective counterparts also hold.*

Proof. In view of (7.1.6), Part (1) is a consequence of Proposition 9.1.3 below. Part (2) follows from Theorem 8.1.3, applied to ψ and $\varphi\circ\psi$. □

We introduce notation that will be used in several arguments.

9.1.2. Let $\mathbf{u}$ be a minimal generating set of $\mathfrak{m}$ modulo $\mathfrak{l}R$ and $\mathbf{x}$ a minimal generating set of $\mathfrak{n}$ modulo $\mathfrak{m}S$, and set $\mathbf{y} = \varphi(\mathbf{u}) \sqcup \mathbf{x}$. The following number is nonnegative:

$$d = \operatorname{edim}\psi + \operatorname{edim}\varphi - \operatorname{edim}(\varphi\circ\psi).$$

Indeed, the isomorphism $\mathfrak{n}/\mathfrak{m}S \cong (\mathfrak{n}/\mathfrak{l}S)/(\mathfrak{m}S/\mathfrak{l}S)$ implies that the set $\mathbf{y}$ generates $\mathfrak{n}/\mathfrak{l}S$. Thus, $\operatorname{edim}(\varphi\circ\psi) \le \operatorname{edim}\varphi + \operatorname{edim}\psi$; that is to say, $d \ge 0$.

PROPOSITION 9.1.3. *The Poincaré series of* ψ*,* φ*, and* $\varphi\circ\psi$ *satisfy*

$$P_N^{\varphi\circ\psi}(t)\cdot(1+t)^d \preccurlyeq P_N^{\varphi}(t)\cdot P_R^{\psi}(t).$$

Proof. In the derived category of S, the isomorphism

$$\mathrm{K}[\mathbf{y};N] \simeq \mathrm{K}[\mathbf{u};R]\otimes_R \mathrm{K}[\mathbf{x};N]$$

combined with the associativity formula for derived tensor products yields

$$\left(h\otimes^{\mathbf{L}}_Q \mathrm{K}[\mathbf{u};R]\right)\otimes^{\mathbf{L}}_R \mathrm{K}[\mathbf{x};N] \simeq h\otimes^{\mathbf{L}}_Q \mathrm{K}[\mathbf{y};N].$$

Thus, one has a standard spectral sequence

$$E^2_{pq} = \operatorname{Tor}^R_p\left(\operatorname{Tor}^Q_q(h,\mathrm{K}[\mathbf{u};R]),\mathrm{K}[\mathbf{x};N]\right) \Rightarrow \operatorname{Tor}^Q_{p+q}(h,\mathrm{K}[\mathbf{y};N]).$$

As the R-module $\operatorname{Tor}^Q(h, \mathrm{K}[\mathbf{w}; R])$ is annihilated by $\mathfrak{m}$, one has an isomorphism

$$\operatorname{Tor}^R_p\left(\operatorname{Tor}^Q_q(h, \mathrm{K}[\mathbf{u}; R]), \mathrm{K}[\mathbf{x}; N]\right) \cong \operatorname{Tor}^R_p(k, \mathrm{K}[\mathbf{x}; N]) \otimes_k \operatorname{Tor}^Q_q(h, \mathrm{K}[\mathbf{u}; R]).$$

The desired inequality is a formal consequence of this isomorphism and the convergence of the spectral sequence; see the proof of Theorem 6.1.1. □

9.2. Upper bound for right factors. When ψ is surjective and $\varphi = \mathrm{id}^R$ the following theorem specializes to a known result for $P^R_N(t)$, see [6, (3.3.2)].

PROPOSITION 9.2.1. *If $\psi\colon Q \to R$ is a local homomorphism, then*

$$P^{\varphi}_N(t) \preccurlyeq P^{\varphi\circ\psi}_N(t) \cdot \frac{1}{1 - t\left(P^{\psi}_R(t) - 1\right)} \cdot (1+t)^{\operatorname{edim} R + \operatorname{edim}\varphi - \operatorname{edim} S}.$$

A similar inequality also holds for Bass series: it is obtained from the one above by replacing $P^{\varphi}_N(t)$ and $P^{\varphi\circ\psi}_N(t)$ with $I^N_{\varphi}(t)$ and $I^N_{\varphi\circ\psi}(t)$, respectively.

Remark. As $P^{\psi}_R(t)$ is a formal power series, the formal power series $1 - t\left(P^{\psi}_R(t) - 1\right)$ has an inverse, which is itself a formal power series. It is given by the formula

$$\frac{1}{1 - t\left(P^{\psi}_R(t) - 1\right)} = \sum_{i=0}^{\infty} t^i \left(P^{\psi}_R(t) - 1\right)^i$$

which shows that it has nonnegative integer coefficients and constant term 1.

Proof. In view of Theorem 4.2.2, it suffices to deal with Poincaré series.

Let E be a DG algebra resolution of h over Q, see [6, (2.1.10)]. Form the DG algebra $A = E \otimes_Q \mathrm{K}[\mathbf{u}; R]$ and the DG A-module $Y = A \otimes_R \mathrm{K}[\mathbf{x}; N]$. The augmentations $E \to h$ and $\mathrm{K}[\mathbf{u}; R] \to k$ are morphisms of DG algebras, where h and k are concentrated in degree 0. They yield a morphism of DG algebras $A \to k$.

Note the standard isomorphisms in the derived category of S-modules:

$$k \otimes^{\mathbf{L}}_A Y \simeq k \otimes^{\mathbf{L}}_A \left(A \otimes^{\mathbf{L}}_R \mathrm{K}[\mathbf{x}; N]\right) \simeq k \otimes^{\mathbf{L}}_R \mathrm{K}[\mathbf{x}; N].$$

For each n they induce l-linear isomorphisms $\operatorname{Tor}^A_n(k, Y) \cong \operatorname{Tor}^R_n(k, \mathrm{K}[\mathbf{x}; N])$, hence

$$\sum_{n\in\mathbb{Z}} \operatorname{rank}_l \operatorname{Tor}^A_n(k, Y)\, t^n = P^{\varphi}_N(t).$$

By the choice of E, there is an isomorphism $\mathrm{H}_n(A) \cong \mathrm{Tor}_n^Q(h, \mathrm{K}[\mathbf{u}; R])$ and hence

$$\sum_{n\in\mathbb{Z}} \mathrm{rank}_k \mathrm{H}_n(A)\, t^n = P_R^{\psi}(t).$$

Likewise, $\mathrm{H}_n(Y) \cong \mathrm{Tor}_n^Q(h, \mathrm{K}[\mathbf{y}; N])$, as $Y \cong E \otimes_Q \mathrm{K}[\mathbf{y}; N]$, so we get

$$\sum_{n\in\mathbb{Z}} \mathrm{rank}_l \mathrm{H}_n(Y)\, t^n = P_N^{\varphi\circ\psi}(t) \cdot (1+t)^e$$

from Proposition 4.3.1. Thus, the inequality we seek translates to

$$(*) \qquad \sum_{n\in\mathbb{Z}} \mathrm{rank}_l \mathrm{Tor}_n^A(k, Y)\, t^n \preccurlyeq \frac{\sum_{n\in\mathbb{Z}} \mathrm{rank}_l \mathrm{H}_n(Y)\, t^n}{1 - t\left(\sum_{n\in\mathbb{Z}} \mathrm{rank}_k \mathrm{H}_n(A)\, t^n - 1\right)}.$$

Now, as $\mathrm{H}_n(A) = 0$ for $n < 0$ and $\mathrm{H}_n(Y) = 0$ for $n \ll 0$, there exists a strongly convergent Eilenberg-Moore spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^{\mathrm{H}(A)}(k, \mathrm{H}(Y))_q \Rightarrow \mathrm{Tor}_{p+q}^A(k, Y),$$

see [27, Chapter 7]. In the light of this, there are (in)equalities

$$\mathrm{rank}_l \mathrm{Tor}_n^A(k, Y) \leq \sum_{p+q=n} \mathrm{rank}_l E^2_{pq} = \sum_{p+q=n} \mathrm{rank}_l \mathrm{Tor}_p^{\mathrm{H}(A)}(k, \mathrm{H}(Y))_q.$$

They can be rewritten as an inequality of formal power series

$$\sum_{n\in\mathbb{Z}} \mathrm{rank}_l \mathrm{Tor}_n^A(k, Y)\, t^n \preccurlyeq \sum_{n\in\mathbb{Z}} \left(\sum_{p+q=n} \mathrm{rank}_l \mathrm{Tor}_p^{\mathrm{H}(A)}(k, \mathrm{H}(Y))_q\right) t^n.$$

By using a standard resolution of k over $\mathrm{H}(A)$, one can compute the graded l-vector space $\mathrm{Tor}_p^{\mathrm{H}(A)}(k, \mathrm{H}(Y))$ as the pth homology of a complex of the form

$$\cdots \longrightarrow \mathrm{H}_{\geqslant 1}(A)^{\otimes n} \otimes_k \mathrm{H}(Y) \longrightarrow \cdots \longrightarrow \mathrm{H}_{\geqslant 1}(A) \otimes_k \mathrm{H}(Y) \longrightarrow \mathrm{H}(Y) \longrightarrow 0.$$

There is thus an inequality of formal Laurent series

$$\sum_{n\in\mathbb{Z}} \left(\sum_{p+q=n} \mathrm{rank}_l \mathrm{Tor}_p^{\mathrm{H}(A)}(k, \mathrm{H}(Y))_q\right) t^n \preccurlyeq \frac{\sum_{n\in\mathbb{Z}} \mathrm{rank}_l \mathrm{H}_n(Y)\, t^n}{1 - t\left(\sum_{n\in\mathbb{Z}} \mathrm{rank}_k \mathrm{H}_n(A)\, t^n - 1\right)}.$$

The desired inequality $(*)$ is obtained by combining the inequalities above. □

COROLLARY 9.2.2. *The curvature and the injective curvature of N over φ are finite.*

Proof. By Remark 7.1.2 it suffices to deal with $\operatorname{curv}_{\varphi} N$. Let p denote the characteristic of k, set $Q = \mathbb{Z}_{(p)}$ and let $\psi\colon\ Q \to R$ be the structure map. Since Q is regular, both $P^{\psi}_N(t)$ and $P^{\varphi\circ\psi}_N(t)$ are Laurent polynomials, see (2.5). Thus, the coefficientwise upper bound for $P^{\varphi}_N(t)$ in Proposition 9.2.1 is a formal Laurent series that represents a rational function. Now apply (7.1.6.1) and (7.1.6.3). □

9.3. Complete intersection homomorphisms. When ψ has a "nice" property the invariants of N over $\varphi \circ \psi$ do not differ much from those over φ.

9.3.1. As in [7], we say that ψ is *complete intersection* at $\mathfrak{m}$ if in some Cohen factorization $Q \to (Q', \mathfrak{l}', k) \xrightarrow{\psi'} R$ of $\grave{\psi}$ the ideal $\operatorname{Ker}\psi'$ is generated by a regular set $\mathbf{f}'$. When this is the case, the Comparison Theorem [10, (1.2)] shows that $\operatorname{Ker}\psi''$ is generated by a regular set whenever $\psi''\ddot{\psi}$ is a Cohen factorization of $\grave{\psi}$. If ψ is weakly regular at $\mathfrak{m}$, see (5.4.1), then it is complete intersection at $\mathfrak{m}$.

THEOREM 9.3.2. *If ψ is complete intersection at $\mathfrak{m}$, then*

$$\operatorname{cx}_{\varphi\circ\psi} N \le \operatorname{cx}_{\varphi} N \le \operatorname{cx}_{\varphi\circ\psi} N + \dim Q - \dim R + \operatorname{edim}\psi$$
$$\operatorname{curv}_{\varphi\circ\psi} N \le \operatorname{curv}_{\varphi} N \le \max\{\operatorname{curv}_{\varphi\circ\psi} N, 1\}.$$

Analogous inequalities also hold for inj cx *and* inj curv.

The next result complements Lemma 5.4.3.

LEMMA 9.3.3. *If $f \in Q$ is a regular element and $R = Q/Qf$, then*

$$P^{\varphi\circ\psi}_N(t) \preccurlyeq P^{\varphi}_N(t)\cdot(1+t) \qquad \textit{and} \qquad P^{\varphi}_N(t) \preccurlyeq P^{\varphi\circ\psi}_N(t)\cdot\frac{1}{(1-t^2)},$$

where $\psi\colon\ Q \to R$ is the canonical map. If, moreover, f is in $\mathfrak{l}\operatorname{Ker}(\varphi\circ\psi)$, then

$$P^{\varphi}_N(t) = P^{\varphi\circ\psi}_N(t)\cdot\frac{1}{(1-t^2)}.$$

Proof. Since $P^{\psi}_R(t) = 1 + t$, the inequality on the left follows from Proposition 9.1.3 and the one on the right from Theorem 9.2.1. The proof of [6, (3.3.5.2)] applies verbatim to yield the equality. □

Proof of Theorem 9.3.2. By Theorem 4.2.2 we restrict to projective invariants.

In view of (7.1.1) we may assume that R and S are complete. We use Construction 5.4.4 and adopt the notation introduced there. By (9.3.1) the kernel of the homomorphism ψ' is generated by a regular set $\mathbf{f}'$; clearly, one has

$$\operatorname{card}\mathbf{f}' = \dim Q' - \dim R = (\dim Q + \operatorname{edim}\psi) - \dim R.$$

Since $\psi'' = Q'' \otimes_{Q'} \psi'$, and Q'' is faithfully flat over Q', the image $\mathbf{f}''$ of $\mathbf{f}'$ in Q'' is a regular set of card $\mathbf{f}'$ elements, and generates the kernel of ψ''.

The complexity and the curvature of N over $\varphi \circ \psi$ equal those over $\varphi \circ \psi'$ by Theorem 5.4.2. Thus, to prove the theorem we may assume that ψ is surjective with kernel generated by a regular set. In this case the desired assertions result from repeated applications of the preceding lemma through (7.1.6). □

9.4. Flat base change. We consider complexes of S-modules induced from R.

THEOREM 9.4.1. *If φ is flat and $S/\mathfrak{m}S$ is artinian, then for every homologically finite complex of R-modules M the following equalities hold:*

$$\operatorname{cx}_{\varphi\circ\psi}(S \otimes_R M) = \operatorname{cx}_{\psi} M \quad \textit{and} \quad \operatorname{curv}_{\varphi\circ\psi}(S \otimes_R M) = \operatorname{curv}_{\psi} M.$$

As usual, this follows from more precise relations involving Poincaré series.

PROPOSITION 9.4.2. *Let M be a homologically finite complex of R-modules. If φ is flat and $\mathfrak{n}^{v} \subseteq \mathfrak{m}S$ for some integer v, then*

$$P_M^{\psi}(t) \cdot v^{-e}(1+t)^e \preccurlyeq P_{M\otimes_R S}^{\varphi\circ\psi}(t) \cdot u^{-1}(1+t)^d \preccurlyeq P_M^{\psi}(t) \cdot (1+t)^e,$$

where $u = \ell(S/\mathfrak{m}S)$, $e = \operatorname{edim}\varphi$, and $d = \operatorname{edim} R + e - \operatorname{edim} S$.

Proof. Let $\mathbf{u}$ be a minimal set of generators of $\mathfrak{m}$ modulo $\mathfrak{l}R$ and set

$$X = h \otimes_Q^{\mathbf{L}} \mathrm{K}[\varphi(\mathbf{u}); S \otimes_R M].$$

The isomorphism $\mathrm{K}[\varphi(\mathbf{u}); (S \otimes_R M)] \simeq \mathrm{K}[\mathbf{u}; M] \otimes_R S$ of complexes of S-modules and the flatness of S over R yield isomorphisms

$$\mathrm{H}(X) \cong \operatorname{Tor}^Q(h, \mathrm{K}[\mathbf{u}; M]) \otimes_R S \cong \operatorname{Tor}^Q(h, \mathrm{K}[\mathbf{u}; M]) \otimes_k (S/\mathfrak{m}S).$$

They produce the following equality of Poincaré series

$$P_M^{\psi}(t) \cdot u = \sum_{n\in\mathbb{Z}} \ell_S \mathrm{H}_n(X)\, t^n.$$

Let $\mathbf{x}$ be a minimal set of generators of $\mathfrak{n}$ modulo $\mathfrak{m}$, and note that the set $\mathbf{y} = \varphi(\mathbf{u}) \sqcup \mathbf{x}$ generates $\mathfrak{n}$ modulo $\mathfrak{l}S$. From our hypothesis and (1.5.3) we get

$$\mathfrak{n}^{v} \subseteq \mathfrak{m}S = \mathfrak{l}S + \mathbf{u}S \subseteq \operatorname*{Ann}_{\mathcal{D}(S)}(X).$$

Iterated applications of Lemma 1.2.3 now give

$$\sum_{n\in\mathbb{Z}} \ell_S \mathrm{H}_n(X)\, t^n \cdot v^{-e}(1+t)^e \preccurlyeq \sum_{n\in\mathbb{Z}} \ell_S \mathrm{H}_n(\mathrm{K}\,[\mathbf{x};X])\, t^n \preccurlyeq \sum_{n\in\mathbb{Z}} \ell_S \mathrm{H}_n(X)\, t^n \cdot (1+t)^e.$$

The isomorphism $\mathrm{K}\,[\mathbf{x};X] \simeq h \otimes^{\mathbf{L}}_Q \mathrm{K}\,[\mathbf{y}; S\otimes_R M]$ and Proposition 4.3.1 now yield

$$\sum_{n\in\mathbb{Z}} \ell_S \mathrm{H}_n(\mathrm{K}\,[\mathbf{x};X])\, t^n = P^{\varphi\circ\psi}_{S\otimes_R M}(t)(1+t)^d.$$

Putting together the formulas above we obtain the desired inequalities. □

10. Localization. A basic and elementary result asserts that Betti numbers of finite modules over local rings do not go up under localization: If M is a finite R-module and $\mathfrak{p}$ is a prime ideal of R, then for each $n \in \mathbb{Z}$ there is an inequality $\beta^{R_\mathfrak{p}}_n(M_\mathfrak{p}) \leq \beta^R_n(M)$.

A notion of localization is also available, and has been systematically used, for complexes over local homomorphisms. Namely, for each prime ideal $\mathfrak{q}$ of S the complex $N_\mathfrak{q}$ is homologically finite over the induced local homomorphism $\varphi_\mathfrak{q}\colon R_\mathfrak{p} \to S_\mathfrak{q}$, where $\mathfrak{p} = \mathfrak{q}\cap R$. However, $\beta^{\varphi_\mathfrak{q}}_n(N_\mathfrak{q})$ may exceed $\beta^\varphi_n(N)$, even when φ is flat.

Example 10.1. Let R be a one-dimensional local domain R whose completion $\widehat{R}$ has a minimal prime ideal $\mathfrak{q}$, such $\widehat{R}_\mathfrak{q}$ is not a field; see Ferrand and Raynaud [15, (3.1)]. Set $S = \widehat{R}$, let $\varphi\colon R \to S$ be the completion map, and note that $\mathfrak{q}\cap R = (0)$; thus, $R_{(0)}$ is a field. For the S-module $N = S$ Proposition 5.2.1 then yields

$$\beta^{\varphi_\mathfrak{q}}_1(N_\mathfrak{q}) = \kappa^{S_\mathfrak{q}}_1(N_\mathfrak{q}) > 0 = \beta^\varphi_1(N).$$

Nor is there an analog over maps of the inequality flat $\dim_{R_\mathfrak{p}} M_\mathfrak{p} \leq$ flat $\dim_R M$.

Example 10.2. Let R be a field, let S be a Cohen-Macaulay ring of positive dimension, and let $\mathfrak{q}$ be a non-maximal prime of S. In view of (3.3), one then has

$$\operatorname{proj\,dim}_{\varphi_\mathfrak{q}} S_\mathfrak{q} = -\operatorname{depth} S_\mathfrak{q} > -\operatorname{depth} S = \operatorname{proj\,dim}_\varphi S.$$

Thus, it is noteworthy that asymptotic invariants over homomorphisms localize as expected. Unlike the corresponding result for complexity and curvature over rings, the theorem below needs a fairly involved argument.

THEOREM 10.3. *For every prime ideal* $\mathfrak{q}$ *in* S *there are inequalities*

$$\operatorname{cx}_{\varphi_\mathfrak{q}} N_\mathfrak{q} \leq \operatorname{cx}_\varphi N \quad \text{and} \quad \operatorname{curv}_{\varphi_\mathfrak{q}} N_\mathfrak{q} \leq \operatorname{curv}_\varphi N.$$

In particular, if $\operatorname{proj\,dim}_\varphi N$ *is finite, then so is* $\operatorname{proj\,dim}_{\varphi_\mathfrak{q}} N_\mathfrak{q}$.

Remark. If D is a normalized dualizing complex for S, then an appropriate shift of $D_{\mathfrak q}$ is a normalized dualizing complex for $S_{\mathfrak q}$. Using this fact together with Theorem 4.2.2, one sees that the preceding theorem has a counterpart for injective complexities and curvatures, *provided* S has a dualizing complex. We do not know whether the last condition is necessary.

Once again, the theorem follows from a more precise result on Poincaré series.

PROPOSITION 10.4. *For every prime ideal* $\mathfrak q$ *in* S *there exists a polynomial* $q(t) \in \mathbb{Z}[t]$ *with nonnegative coefficients such that the following inequality holds*

$$P^{\varphi_{\mathfrak q}}_{N_{\mathfrak q}}(t) \preccurlyeq q(t) \cdot P^{\varphi}_{N}(t).$$

Proof. Form a commutative diagram of local homomorphisms

$$\begin{tikzcd} R \arrow[r, "\dot\varphi"] \arrow[d, "\varphi"'] \arrow[dr, "\grave\varphi"] & R' \arrow[d, "\varphi'"] \\ S \arrow[r, "\sigma"'] & \widehat{S} \end{tikzcd}$$

where σ is the completion map of S in the $\mathfrak n$-adic topology and the upper triangle is a minimal Cohen factorization of $\grave\varphi$. By faithful flatness, choose $\widetilde{\mathfrak q}$ in $\operatorname{Spec}\widehat{S}$ so that $\widetilde{\mathfrak q} \cap S = \mathfrak q$ and $\ell\left(\widehat{S}_{\widetilde{\mathfrak q}}/\mathfrak q\widehat{S}_{\widetilde{\mathfrak q}}\right) = u < \infty$. Setting $\widetilde{\mathfrak p} = \widetilde{\mathfrak q} \cap R'$ and $\mathfrak p = \widetilde{\mathfrak q} \cap R$ and localizing, one gets a commutative diagram of local homomorphisms

$$\begin{tikzcd} R_{\mathfrak p} \arrow[r, "\dot\varphi_{\widetilde{\mathfrak p}}"] \arrow[d, "\varphi_{\mathfrak q}"'] \arrow[dr, "\grave\varphi_{\widetilde{\mathfrak q}}"] & R'_{\widetilde{\mathfrak p}} \arrow[d, "\varphi'_{\widetilde{\mathfrak q}}"] \\ S_{\mathfrak q} \arrow[r, "\sigma_{\widetilde{\mathfrak q}}"'] & \widehat{S}_{\widetilde{\mathfrak q}} \end{tikzcd} \quad \text{that we redraw as} \quad \begin{tikzcd} P \arrow[r, "\dot\pi"] \arrow[d, "\varkappa"'] \arrow[dr, "\pi"] & P' \arrow[d, "\pi'"] \\ Q \arrow[r, "\tau"'] & T. \end{tikzcd}$$

The homomorphism τ is flat with artinian closed fiber. Indeed, τ is a localization of the flat homomorphism σ, and its closed fiber is the ring $\widehat{S}_{\widetilde{\mathfrak q}}/\mathfrak q S_{\widetilde{\mathfrak q}}$.

Set $L = N_{\mathfrak q}$. Proposition 9.4.2 yields the inequality

$$P^{\varkappa}_{L}(t) \cdot (1+t)^{s} \preccurlyeq P^{\tau\circ\varkappa}_{L\otimes_Q T}(t) \cdot (1+t)^{e}(\upsilon^{s}/u),$$

where s, e, u, and v are nonnegative integers. As $\tau \circ \varkappa = \pi = \pi' \circ \dot{\pi}$, one has

$$P^{\tau\circ\varkappa}_{L\otimes_Q T}(t) = P^{\pi'\circ\dot{\pi}}_{L\otimes_Q T}(t).$$

Set $p(t) = P^{\dot{\pi}}_{P'}(t)$. Since $\dot{\pi}$ is flat, $p(t)$ is a polynomial: see, for example, (8.2.2). Thus, Proposition 9.1.3 provides a nonnegative integer g such that

$$P^{\pi'\dot{\pi}}_{L\otimes_Q T}(t) \cdot (1+t)^g \preccurlyeq P^{\pi'}_{L\otimes_Q T}(t) \cdot p(t).$$

Next we note that over the ring $T = \widehat{S}_{\widetilde{\mathfrak{q}}}$ there are isomorphisms of complexes

$$L \otimes_Q T \cong (N \otimes_S Q) \otimes_Q T \cong N \otimes_S T \cong (N \otimes_S \widehat{S}) \otimes_{\widehat{S}} T = \widehat{N} \otimes_{\widehat{S}} T \cong \widehat{N}_{\widetilde{\mathfrak{q}}}.$$

They give the first equality in the sequence below, where the second one comes from (4.1.2), the last from Remark 4.1.4, and the inequality is classical:

$$P^{\pi'}_{L\otimes_Q T}(t) = P^{\pi'}_{\widehat{N}_{\widetilde{\mathfrak{q}}}}(t) = P^{R_{\widetilde{\mathfrak{p}}}}_{\widehat{N}_{\widetilde{\mathfrak{q}}}}(t) \preccurlyeq P^{R'}_{\widehat{N}}(t) = P^{\varphi}_{N}(t).$$

Putting together the comparisons above, one obtains that

$$P^{\varkappa}_{L}(t) \cdot (1+t)^{(s+g)} \preccurlyeq P^{\varphi}_{N}(t) \cdot p(t)(v^s/u).$$

The inequality we seek is contained in the one above, because $1 \preccurlyeq (1+t)^{(s+g)}$. $\square$

11. Extremality. A recurrent theme in local algebra is that the homological properties of the R-module k carry a lot of information on the structure of R. Thus, modules or complexes "homologically similar" to k provide test objects for properties of R. We focus on rates of growth of Betti numbers, which controls regularity and the complete intersection property. As in [5], we say that a finite R-module is (*injectively*) *extremal* if its (injective) complexity and curvature are equal to those of k.

Proposition 7.1.3 shows that these same numbers are also upper bounds for (injective) complexities and curvatures over φ. This leads to obvious extensions of the notions of extremality. One reason to study them may not be obvious a priori: it provides many new classes of extremal modules over R, see Remark 11.4.

We say that the complex N is *extremal over* φ if

$$\operatorname{cx}_\varphi N = \operatorname{cx}_R k \quad \text{and} \quad \operatorname{curv}_\varphi N = \operatorname{curv}_R k.$$

It is *injectively extremal over* φ if

$$\operatorname{inj\,cx}_\varphi N = \operatorname{cx}_R k \quad \text{and} \quad \operatorname{inj\,curv}_\varphi N = \operatorname{curv}_R k.$$

Using $\operatorname{inj} \operatorname{cx}_R k$ and $\operatorname{inj} \operatorname{curv}_R k$ to define injective extremality yields the same result: computing with a minimal free resolution of k, one gets $\mu_R^n(k) = \beta_n^R(k)$.

The homological characterization of complete intersections in Theorem 7.1.5 shows that the notion of extremality has two distinct aspects.

Remark 11.1. If R is complete intersection, then N is (injectively) extremal over φ if and only if its (injective) complexity is equal to $\operatorname{codim} R$.

If R is not complete intersection, then N is (injectively) extremal over φ if and only if its (injective) curvature is equal to $\operatorname{curv}_R k$.

Remark 11.2. The definition of separation and (7.1.6) yield: If N is (injectively) separated over φ and $\mathrm{H}(N) \neq 0$, then it is (injectively) extremal over φ.

Separation is a much stronger condition than extremality:

Example 11.3. Assume that R is not regular and set $\varphi = \mathrm{id}^R$.

(1) If R is complete intersection of codimension ≥ 2, and M_n is the nth syzygy of an extremal R-module M, then it is clear that each M_n is extremal; however, it follows from [11, (6.2)] that M_n can be separated for at most one value of n.

(2) Remark 11.5 below shows that $k \oplus R$ is extremal, but not separated.

Remark 11.4. When N is homologically finite over R, its Betti numbers and Bass numbers may be easier to compute over φ than over R. On the other hand, Theorem 7.2.3 shows that the corresponding asymptotic invariants over R and over φ coincide. This can be used to identify new classes of extremal complexes over R. For example, Remark 11.2 shows that if the local homomorphism $\varphi\colon R \to S$ is module finite, the ring S is regular, and $\mathrm{H}(N) \neq 0$, then N is extremal over R.

Remark 11.5. Let X be a homologically finite complex of S-modules. The complex of S-modules $N \oplus X$ is separated over φ if and only if both N and X are separated, while it is extremal over φ if and only if one of N or X is extremal.

Indeed, the claim on extremality is clear. The claim on separation is verified by using Proposition 6.1.1 along with the equalities

$$K^S_{N\oplus X}(t) = K^S_N(t) + K^S_X(t) \quad \text{and} \quad P^\varphi_{N\oplus X}(t) = P^\varphi_N(t) + P^\varphi_X(t).$$

The main result in this section quantifies and significantly generalizes a theorem of Koh and Lee, see [22, (2.6.i)]. The idea to use socles to locate nonzero homology classes is inspired by an argument in their proof of [22, (1.2.i)].

Let L be an S-module. Recall that its *socle* is the S-submodule

$$\operatorname{Soc}_S(L) = \{a \in L \mid \mathfrak{n}a = 0\}.$$

Note that $\operatorname{Soc}_R(L)$ is also defined; it is an S-submodule of L and contains $\operatorname{Soc}_S(L)$.

THEOREM 11.6. *Let L be a module. If $\mathbf{v}$ is an L-regular set in S such that*
(a) $\operatorname{Soc}_S(L/\mathbf{v}L) \nsubseteq \mathfrak{m}(L/\mathbf{v}L)$, *or*
(b) $\operatorname{Soc}_R(L/\mathbf{v}L) \nsubseteq \mathfrak{m}(L/\mathbf{v}L)$ *and the ring $S/\mathfrak{m}S$ is artinian,*
then L is extremal and injectively extremal over φ.

The proof is based on a simple sufficient condition for extremality.

LEMMA 11.7. *If X is a homologically finite complex of S-modules such that*

$$0 < \ell_S\left(\mathrm{H}(X)/\mathfrak{m}\mathrm{H}(X)\right) < \infty \quad \textit{and} \quad \operatorname{Soc}_R\left(\operatorname{Ker}(\partial^X_j)\right) \nsubseteq \mathfrak{m}X_j + \partial(X_{j+1})$$

for some $j \in \mathbb{Z}$, then for all $n \in \mathbb{Z}$ the following inequalities hold

$$\ell_S\left(\operatorname{Tor}^R_n(k,X)\right) \geq \beta^R_{n-j}(k) \quad \textit{and} \quad \ell_S\left(\operatorname{Ext}^n_R(k,X)\right) \geq \beta^R_{n+j}(k).$$

In particular, the complex X is extremal and injectively extremal over ϕ^i.

Proof. Set $X'_n = X_n$ for all $n \neq j$ and $X'_j = \mathfrak{m}X_j + \partial(X_{j+1})$. It is easy to see that X' is a subcomplex of X. We form a commutative diagram

$$\begin{tikzcd}
\Sigma^j \operatorname{Soc}_R\left(\operatorname{Ker}(\partial^X_j)\right) = V \arrow[r, "\pi"] \arrow[d, "\zeta"'] & W = \dfrac{V}{V \cap X'} \arrow[d, "\iota"] \\
X \arrow[r, "\rho"] & Y = \dfrac{X}{X'}
\end{tikzcd}$$

of complexes of S-modules, where V, W, and Y are concentrated in degree j. The ring R acts on them through k, so $\operatorname{Tor}^R(k,-)$ applied to the diagram above yields a commutative diagram of graded S-modules

$$\begin{tikzcd}
\operatorname{Tor}^R(k,k) \otimes_k V \arrow[r, "\operatorname{Tor}^R(k,k)\otimes_k \pi"] \arrow[d, "\operatorname{Tor}^R(k,\zeta)"'] & \operatorname{Tor}^R(k,k) \otimes_k W \arrow[d, "\operatorname{Tor}^R(k,k)\otimes_k \iota"] \\
\operatorname{Tor}^R(k,X) \arrow[r, "\operatorname{Tor}^R(k,\rho)"] & \operatorname{Tor}^R(k,k) \otimes_k Y.
\end{tikzcd}$$

By construction, the map π is surjective and the map ι is injective, so the image of $\operatorname{Tor}^R(k,\rho)$ contains an isomorphic copy of $\operatorname{Tor}^R(k,k) \otimes_k W$. Thus,

$$\ell_S\left(\operatorname{Tor}^R_n(k,X)\right) \geq \beta^R_{n-j}(k) \cdot \ell_S(W)$$

for all $n \in \mathbb{Z}$. Since $W \not\simeq 0$, by hypothesis, extremality follows from Remark 7.2.4.

Similar arguments yield the statements concerning injective invariants. □

Proof of Theorem 11.6. Corollary 7.2.2 shows that L and $L/\mathbf{v}L$ are extremal simultaneously, so replacing L with $L/\mathbf{v}L$, we assume one of the conditions:

(a) $\operatorname{Soc}_S(L) \not\subseteq \mathfrak{m}L$, or

(b) $\operatorname{Soc}_R(L) \not\subseteq \mathfrak{m}L$ and $S/\mathfrak{m}S$ is artinian.

(a) Let $\mathbf{y}$ be a finite set of s generators of $\mathfrak{n}$, and set $X = \mathrm{K}[\mathbf{y}; L]$. Note that $\operatorname{rank}_l \mathrm{H}(X)$ is finite, that $\operatorname{Ker}(\partial_s^X) = \operatorname{Soc}_S(L)$, and that $X_{s+1} = 0$. Thus, Lemma 11.7 shows that X is extremal and Proposition 7.2.1 completes the proof.

(b) Apply Lemma 11.7 to the complex $X = L$. □

We wish to compare the hypotheses of various theorems yielding extremality.

Example 11.8. Let $R = S = k[x]/(x^3)$ where k is a field of characteristic 2, let ϕ_R be the Frobenius endomorphism of R, and set $L = S$. One then has

$$\operatorname{Soc}_R(L) = (x), \qquad \operatorname{Soc}_S(L) = (x^2) = \mathfrak{m}L \qquad \text{and} \qquad \operatorname{spread}_S L = 4,$$

see Remark 3.9. Thus, Theorem 11.6.b shows that S is extremal over φ, but neither Theorem 11.6.a nor Theorem 6.2.2 can be applied.

Remark 11.9. The conditions "extremal" and "injectively extremal" are independent in general, even over $\varphi = \mathrm{id}^R$. For instance, Example 7.1.4 yields

$$\operatorname{cx}_R k = \operatorname{cx}_R E = \operatorname{inj\,cx}_R R = \infty$$
$$\operatorname{curv}_R k = \operatorname{curv}_R E = \operatorname{inj\,curv}_R R = 2$$

for $R = k[x,y]/(x^2, xy, y^2)$ and the injective hull E of k over R. Thus, E is extremal but not injectively extremal, while R is injectively extremal, but not extremal.

12. Endomorphisms. Let ϕ be a local endomorphism, that is, a local homomorphism

$$\phi\colon (R, \mathfrak{m}, k) \to (R, \mathfrak{m}, k)$$

and let N be a homologically finite complex of R-modules.

12.1. Contractions. We say that the homomorphism ϕ is *contracting*, or that ϕ is a *contraction*, if for each element x in $\mathfrak{m}$ the sequence $(\phi^i(x))_{i\geqslant 1}$ converges to 0 in the $\mathfrak{m}$-adic topology. Observe that ϕ is contracting if ϕ^j is contracting for some integer $j \geq 1$, and only if ϕ^j is contracting for every integer $j \geq 1$.

One motivation for considering contracting homomorphisms comes from:

Example 12.1.1. The archetypal contraction is the *Frobenius endomorphism* ϕ_R of a ring R of prime characteristic p, defined by $\phi_R(x) = x^p$ for all $x \in R$.

Interesting contractions are found in every characteristic:

Example 12.1.2. Let k be a field, let G be an additive semigroup without torsion, let $k[G]$ denote the semigroup ring of G over k, and let $\mathfrak{p}$ denote the maximal ideal of $k[G]$ generated by the elements of G. For each nonnegative integer s, the endomorphism σ of the semigroup G, given by $\sigma(g) = s \cdot g$, defines an endomorphism $k[\sigma]$ of the ring $k[G]$. It satisfies $k[\sigma](\mathfrak{p}) \subseteq \mathfrak{p}$, and so it induces an endomorphism ϕ of the local ring $R = k[G]_{\mathfrak{p}}$. The endomorphism ϕ is contracting when $s \geq 2$.

Many complexes are separated over contractions. Theorem 6.2.2 implies the next result, which uses the homotopy Loewy length $\ell\ell_{\mathcal{D}(S)}(N)$ defined in (6.2).

THEOREM 12.1.3. *Assume ϕ is a contraction and* $\mathrm{H}(N) \neq 0$. *If $j \geq 1$ and $q \geq 2$ satisfy $\phi^j(\mathfrak{m}) \subseteq \mathfrak{m}^q$, then N is separated and injectively separated over ϕ^i for all*

$$i \geq j \log_q (\ell\ell_{\mathcal{D}(S)}(\mathrm{K}^S [N])).$$

In particular, N is extremal and injectively extremal over ϕ^i.

The following alternative description of contractions shows that the numbers j and q in the hypothesis of the theorem always exist, and gives bounds for them. Example 12.1.6 shows that these bounds cannot be improved in general.

LEMMA 12.1.4. *The map ϕ is contracting if and only if $\phi^{\operatorname{edim} R}(\mathfrak{m}) \subseteq \mathfrak{m}^2$.*

Proof. The "if" part is clear, so we assume ϕ is contracting.

Let $\delta\colon \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$ be the map induced by ϕ. It is a homomorphism of abelian groups, which defines on $\mathfrak{m}/\mathfrak{m}^2$ a filtration

$$0 = \operatorname{Ker}(\delta^0) \subseteq \operatorname{Ker}(\delta^1) \subseteq \cdots \subseteq \operatorname{Ker}(\delta^i) \subseteq \operatorname{Ker}(\delta^{i+1}) \subseteq \cdots .$$

We need to prove that for $r = \operatorname{edim} R$ one has $\operatorname{Ker}(\delta^r) = \mathfrak{m}/\mathfrak{m}^2$, that is, $\delta^r = 0$.

Each subgroup $\operatorname{Ker}(\delta^i)$ is a k-vector subspace of $\mathfrak{m}/\mathfrak{m}^2$, by a direct verification. We show next that if $\operatorname{Ker}(\delta^i) = \operatorname{Ker}(\delta^{i+1})$ for some $i \geq 1$, then $\operatorname{Ker}(\delta^i) = \operatorname{Ker}(\delta^j)$ for all $j \geq i$. By induction, it suffices to do it for $j = i + 2$. If $\delta^{i+2}(x) = 0$, then

$$\delta^{i+2}(x) = \delta^{i+1}(\delta(x)) = 0$$

implies $\delta(x) \in \operatorname{Ker}(\delta^{i+1}) = \operatorname{Ker}(\delta^i)$. Therefore, $\delta^{i+1}(x) = 0$, as desired.

Since $\operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2 = r$, the properties of $\operatorname{Ker}(\delta^i)$ that we have established imply equalities $\operatorname{Ker}(\delta^j) = \operatorname{Ker}(\delta^r)$ for all $j \geq r$, that is,

$$\operatorname{Ker}(\delta^r) = \bigcup_{i=0}^{\infty} \operatorname{Ker}(\delta^i).$$

Our hypothesis means that the union above is all of $\mathfrak{m}/\mathfrak{m}^2$. □

We can do better than Theorem 12.1.3 when $N = R$, at least for curvature.

THEOREM 12.1.5. *If $\phi\colon R \to R$ is a contracting endomorphism, then*

$$\operatorname{curv}_\phi R = \operatorname{curv}_R k.$$

If, in addition, the ring R is Gorenstein, then

$$\operatorname{inj\,curv}_\phi R = \operatorname{curv}_R k.$$

Proof. By the preceding theorem, there is an integer j giving the equality below:

$$\operatorname{curv}_R k = \operatorname{curv}_{\phi^j} R \leq \operatorname{curv}_\phi R \leq \operatorname{curv}_R k.$$

The inequalities come from Theorem 9.1.1.1 and from Proposition 7.1.3.5.

If R is Gorenstein, then $R^\dagger \simeq R$, see (3.5), hence the middle equality below:

$$\operatorname{inj\,curv}_\phi R = \operatorname{curv}_\phi R^\dagger = \operatorname{curv}_\phi R = \operatorname{curv}_R k.$$

The other two are given by Remark 7.1.2 and the first part of the theorem. □

Theorem 12.1.5 has no counterpart for complexities.

Example 12.1.6. Let k be a field, set $R = k[[x_1, \dots, x_r]]/(x_1^2, \dots, x_r^2)$, and let $\phi\colon R \to R$ be the k-algebra homomorphism given by

$$\phi(x_1) = 0 \quad \text{and} \quad \phi(x_i) = x_{i-1} \quad \text{for} \quad 2 \leq i \leq r.$$

Then $\operatorname{cx}_{\phi^j} R = \min\{j, r\}$ for each integer $j \geq 0$.

Indeed, fix an integer $j \geq 1$ and form the ring

$$S = k[[x_{j+1}, \dots, x_r]]/(x_{j+1}^2, \dots, x_r^2).$$

The map ϕ^j factors as $R \xrightarrow{\pi} S \xrightarrow{\iota} R$, where π is the canonical surjection with kernel $(x_1, \dots, x_j)$ and ι is the k-algebra homomorphism with $\iota(x_i) = x_{i-j}$ for

$j < i \leq r$. As ι is flat and R is artinian, Theorem 9.4.1 yields $\operatorname{cx}_{\phi^j} R = \operatorname{cx}_\pi S$. Set

$$Q = k[[x_1, \dots, x_r]]/(x_{j+1}^2, \dots, x_r^2)$$

and let $\psi\colon Q \to R$ be the canonical surjection with kernel $(x_1^2, \dots, x_j^2)$. With $m = \min\{j, r\}$, repeated application of Lemma 9.3.3 yields the first equality in the sequence

$$P_S^\pi(t) = \frac{P_S^{\pi\circ\psi}(t)}{(1-t^2)^m} = \frac{P_S^Q(t)}{(1-t^2)^m} = \frac{(1+t)^m}{(1-t^2)^m} = \frac{1}{(1-t)^m}.$$

The second equality by Remark 4.1.2; the third holds because $\mathrm{K}[\{x_1, \dots, x_m\}; Q]$ is a minimal resolution of S over Q. Now invoke (7.1.6).

12.2. Frobenius endomorphisms. In this subsection R has prime characteristic p and $\phi_R\colon R \to R$ is its Frobenius endomorphism. This is a contracting endomorphism, so the results from the preceding subsection apply. One noteworthy additional feature is that they can be interpreted entirely in terms of classically defined invariants. Indeed, Theorem 7.2.3 validates the following:

Remark 12.2.1. Let ${}^{i}\!N$ denote N as an R-R-bimodule with the left action through ϕ_R^i and the right action the usual one. Each $\operatorname{Tor}_n^R(k, {}^{i}\!N)$ is an R-module where the action of R is induced from the right action of R on ${}^{i}\!N$; by (1.4.2) and Lemma 1.3.2 it has finite length.

For each integer $i \geq 1$ the following equalities hold:

$$\begin{aligned}
\operatorname{cx}_{\phi_R^i} N &= \inf\left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a number } c \in \mathbb{R} \text{ such that} \\ \ell_R \operatorname{Tor}_n^R(k, {}^{i}\!N) \leq c n^{d-1} \text{ for all } n \gg 0 \end{array} \right\} \\
\operatorname{curv}_{\phi_R^i} N &= \limsup_n \sqrt[n]{\ell_R \operatorname{Tor}_n^R(k, {}^{i}\!N)}.
\end{aligned}$$

This subsection is organized around the following:

Question 12.2.2. Is N separated (respectively, extremal) over ϕ_R^i for all $i \geq 1$?

A lot of evidence points to a positive answer.

Remark 12.2.3. When R is not complete intersection, Theorem 12.1.3 shows that N is separated and injectively separated over ϕ_R^i for all $i \geq \log_p(\ell\ell_{\mathcal{D}(R)}(N))$.

When R is complete intersection $P_k^R(t) = (1+t)^d/(1-t)^c$, see (6.3.6.2), so the next theorem asserts that N is separated and injectively separated over ϕ_R^i.

THEOREM 12.2.4. *If R is complete intersection, $d = \dim R$, and $c = \operatorname{codim} R$, then for each $i \geq 1$ the following equalities hold:*

$$P_N^{\phi_R^i}(t) = K_N^R(t) \cdot \frac{(1+t)^d}{(1-t)^c} \qquad \text{and} \qquad I^N_{\phi_R^i}(t) = K_N^R(t) \cdot \frac{(1+t)^d t^c}{(1-t)^c}.$$

In particular, N is extremal and injectively extremal over ϕ_R^i.

COROLLARY 12.2.5. *For every ring R of positive characteristic, the module R is extremal over ϕ_R^i for each integer $i \geq 1$.*

Proof. In view of Remark 11.1, Theorem 12.1.5 establishes the assertion when R is not complete intersection. When it is, Theorem 12.2.4 applies. □

The proof of the theorem is given at the end of this section. We approach it through an explicit description of minimal Cohen factorizations of powers of the Frobenius endomorphisms of complete rings.

Construction 12.2.6. Let $\mathbf{v} = \{v_1, \dots, v_r\}$ be a minimal set of generators for $\mathfrak{m}$. Identifying R with its image in $\widehat{R}$ under the completion map, note that $\mathbf{v}$ minimally generates the maximal ideal of $\widehat{R}$.

Let $\mathbf{x} = \{x_1, \dots, x_r\}$ be a set of formal indeterminates and set $Q = k[[\mathbf{x}]]$. With the standard abbreviation $\mathbf{x}^J = x_1^{j_1} \cdots x_r^{j_r}$ for $J = (j_1, \dots, j_r)$, each $g \in Q$ has a unique expression $g = \sum_{J \in \mathbb{N}^r} a_J \mathbf{x}^J$ with $a_J \in k$. For each positive integer q set

$$g^{[q]} = \sum_{J \in \mathbb{N}^r} a_J^q \mathbf{x}^J.$$

Choose, by Cohen's Structure theorem, a surjective homomorphism $\psi\colon Q \to \widehat{R}$, such that $\psi(x_j) = v_j$ for $j = 1, \dots, r$. Let $\mathbf{f} = \{f_1, \dots, f_c\}$ be a minimal generating set of $\operatorname{Ker}\psi$. The choices made so far imply $\mathbf{f} \subseteq (\mathbf{x})^2$. Set

$$\mathbf{f}^q = \{f_1^q, \dots, f_c^q\} \qquad \text{and} \qquad \mathbf{f}^{[q]} = \{f_1^{[q]}, \dots, f_c^{[q]}\}.$$

Fix an integer $i \geq 1$, set $q = p^i$ and $\phi = \phi_{\widehat{R}}^i$. Let $\mathbf{y} = \{y_1, \dots, y_r\}$ denote a second family of formal indeterminates. With the data above, form the diagram

$$
\begin{tikzcd}
k[[\mathbf{x}]] \arrow[r, equal] & Q \arrow[rrr, "\phi^i_Q"] \arrow[dd, "\psi"'] & & & Q \arrow[r, equal] \arrow[d, "\psi'"] & k[[\mathbf{x}]] \\
& & \frac{k[[\mathbf{x}]]}{(\mathbf{f}^{[q]})}[[\mathbf{y}]] \arrow[r, equal] & R' \arrow[r, "\rho"] \arrow[dr, "\phi'"] & R'' \arrow[r, equal] \arrow[d, "\phi''"] & \frac{k[[\mathbf{x}]]}{(\mathbf{f}^q)} \\
\frac{k[[\mathbf{x}]]}{(\mathbf{f})} \arrow[r, equal] & \widehat{R} \arrow[urr, "\dot{\phi}"] \arrow[rrr, "\phi"] & & & \widehat{R} \arrow[r, equal] & \frac{k[[\mathbf{x}]]}{(\mathbf{f})}
\end{tikzcd}
$$

of local homomorphisms, where the new objects are defined as follows.

The rings R' and R'' are described by the respective equalities.

The maps ψ' and ϕ'' are the canonical surjections.

The maps ϕ^i_Q and $\dot{\phi}$ are given by the formulas

$$\phi^i_Q(g) = g^q \qquad \text{and} \qquad \dot{\phi}\,(g + (\mathbf{f})) = g^{[q]} + (\mathbf{f}^{[q]}).$$

The map ρ is the unique homomorphism of complete k-algebras satisfying

$$\rho(x_i + (\mathbf{f}^{[q]})) = x_i^q + (\mathbf{f}^q) \qquad \text{and} \qquad \rho(y_i) = x_i + (\mathbf{f}^q)$$

for $i = 1, \dots, r$; note that ρ is surjective.

The map ϕ' is the composition $\phi'' \circ \rho$.

The definitions above show that the diagram commutes.

PROPOSITION 12.2.7. *The maps in Construction 12.2.6 have the properties below.*

(1) $\phi'\dot{\phi}$ *is a minimal Cohen factorization of* ϕ.

(2) *There is an equality of formal Laurent series*

$$P^{\phi}_N(t) = P^{\phi''}_{\widehat{N}}(t) \cdot (1+t)^r.$$

Proof. (1) Since $\dot{\phi}(x_i) = x_i$ for $i = 1, \dots, r$, the ring $R'/\mathfrak{m}R'$ is isomorphic to the regular ring $k[[\mathbf{y}]]$. To prove that $\dot{\phi}$ is flat, consider the composition

$$Q = k[[\mathbf{x}]] \xrightarrow{\phi^i_Q} k[[\mathbf{x}]] \xrightarrow{\iota} k[[\mathbf{x}, \mathbf{y}]] = Q',$$

where ι is the natural inclusion. Thus, $\iota \circ \phi^i_Q$ maps the Q-regular set $\mathbf{x}$ to the Q'-regular set $\mathbf{x}^q$. Computing $\operatorname{Tor}^Q(k, Q')$ from the resolution $\mathrm{K}[\mathbf{x}; Q]$ of k over Q, one gets $\beta^Q_n(Q') = 0$ for $n > 0$. It follows that Q' is flat over Q, see (2.2). As $\dot\phi$ is obtained from $\iota \circ \phi^i_Q$ by base change along ψ, we conclude that $\dot\phi$ is flat, as desired.

As ϕ' is surjective, $\phi = \phi'\dot\phi$ is a Cohen factorization.

(2) The kernel of ρ is generated by the set $\{x_1 - y_1^q, \ldots, x_r - y_r^q\}$, which is regular and superficial. Remark 4.1.4 and Theorem 5.4.3 yield

$$P^\phi_N(t) \;=\; P^{\phi'}_{\widehat{N}}(t) = P^{\phi''}_{\widehat{N}}(t) \cdot (1+t)^r. \qquad \square$$

Proof of Theorem 12.2.4. We use Construction 12.2.6. As R is complete intersection, the set $\mathbf{f}$ is regular, see (5.3.1). It follows that so is $\mathbf{f}^q$; note that $\mathbf{f}^q$ is contained in $\mathfrak{l}(\mathbf{f}Q)$, where $\mathfrak{l}$ is the maximal ideal of Q. Lemma 12.2.7.2, Proposition 9.3.3, and Example 6.1.3 provide the equalities below

$$\begin{aligned} P^\phi_N(t) \;&=\; P^{\phi''}_{\widehat{N}}(t) \cdot (1+t)^r \\ &=\; P^{\phi''\circ\psi'}_{\widehat{N}}(t) \cdot \frac{1}{(1-t^2)^c} \cdot (1+t)^r \\ &=\; P^Q_k(t) \cdot \frac{K^R_N(t)}{(1+t)^r} \cdot \frac{(1+t)^d}{(1-t)^c} \\ &=\; K^R_N(t) \cdot \frac{(1+t)^d}{(1-t)^c}. \end{aligned}$$

Theorem 4.2.2 yields the desired expression for $I^N_\varphi(t)$. $\square$

13. Local homomorphisms. In this section we use the techniques and results developed earlier in the paper to study relations between the ring theoretical properties of R and S and the homological properties of the R-module S. First we look at descent problems.

THEOREM 13.1. *Let $\varphi\colon R \to S$ be a local homomorphism and let N be a homologically finite complex of S-modules with* $\mathrm{H}(N) \neq 0$.

(1) *If* flat $\dim_R N < \infty$ *and S is regular, then so is R.*

(2) *If* $\operatorname{curv}_\varphi S \leq 1$ *and S is complete intersection, then so is R.*

Remark. Part (1) of the theorem is due to Apassov [2, Theorem R]. Part (2) significantly generalizes [7, (5.10)], where it is proved that maps of finite flat dimension descend the complete intersection property. Dwyer, Greenlees, and Iyengar [14] show that this property descends even under the weaker hypothesis $\operatorname{curv}_\varphi N \leq 1$.

Proof. (1) One has $\beta_n^{\varphi}(N) = 0$ for $n \gg 0$, see (1.2.2). As N is separated by Corollary 6.2.3, the equality in (6.1.2) implies $\beta_n^R(k) = 0$ for all $n \gg 0$.

(2) Applying Theorem 9.1.1.1 to the composition $R \to S \xrightarrow{=} S$ and the S-module l, we obtain the first inequality below:

$$\operatorname{curv}_{\varphi} l \leq \max\{\operatorname{curv}_{\varphi} S, \operatorname{curv}_{\mathrm{id}^S} l\} \leq \max\{\operatorname{curv}_{\varphi} S, 1\} \leq 1.$$

The second one comes from Theorem 7.1.5, the third from our hypothesis. We conclude that R is complete intersection by referring once more to (7.1.5). □

Next we extend to arbitrary local homomorphisms a characterization of Gorenstein rings, due to Peskine and Szpiro in the case of surjective maps.

THEOREM 13.2. *For a local homomorphism φ: $R \to S$ the condition* inj $\dim_R S < \infty$ *holds if and only if* flat $\dim_R S < \infty$ *and the ring R is Gorenstein.*

Proof. Over a Gorenstein ring the flat dimension of a module is finite if and only if its injective dimension is, see [25, (2.2)], so we have to prove that inj $\dim_R S < \infty$ implies R is Gorenstein. Let $R \to R' \to \widehat{S}$ be a minimal Cohen factorization of $\acute{\varphi}$. Corollary 2.5 gives inj $\dim_{R'} \widehat{S} < \infty$. As $R' \to \widehat{S}$ is surjective, R' is Gorenstein by Peskine and Szpiro [29, (II.5.5)]. By flat descent, see [26, (23.4)], so is R. □

Finally, we turn to properties of a local ring R equipped with a contracting endomorphism ϕ, for instance, a ring of prime characteristic with its Frobenius endomorphism. Some of our theorems are stated in terms of homological properties of the R-module ${}^{\phi^i}R$, that is, R viewed as a module over itself through ϕ^i.

The prototype of such results is a famous theorem of Kunz, [23, (2.1)]: A ring R of prime characteristic is regular if ${}^{\phi^i}R$ is flat for some $i \geq 1$, only if ${}^{\phi^i}R$ is flat for all i. Later, Rodicio [32, Theorem 2] showed that the flatness hypothesis on ${}^{\phi^i}R$ can be replaced by one of finite flat dimension. Our first criterion extends these results to all contracting endomorphisms and provides tests for regularity by finite injective dimension, which are new even for the Frobenius endomorphism:

THEOREM 13.3. *For a contraction ϕ: $R \to R$ the following are equivalent.*

(i) *R is regular.*

(ii) flat $\dim_R {}^{\phi^i}R = \dim R/(\phi^i(\mathfrak{m})R)$ *for all integers $i \geq 1$.*

(iii) flat $\dim_R {}^{\phi^i}R < \infty$ *for some integer $i \geq 1$.*

(iv) inj $\dim_R {}^{\phi^i}R = \dim R$ *for all integers $i \geq 1$.*

(v) inj $\dim_R {}^{\phi^i}R < \infty$ *for some integer $i \geq 1$.*

Proof. The implication (i) ⇒ (iv) holds by Theorem 2.1, the implication (iv) ⇒ (v) is clear, while (v) ⇒ (iii) comes from Theorem 13.2.

(iii) $\Rightarrow$ (i). Since ϕ^i is a contraction, from Theorem 12.1.5 one gets $\operatorname{curv}_R k = \operatorname{curv}_{\phi^i} R = 0$. This means $\operatorname{proj\,dim}_R k$ is finite, that is, R is regular.

(i) $\Rightarrow$ (ii). Let $\mathbf{v}$ be a minimal set of generators of $\mathfrak{m}$. The Koszul complex $\mathrm{K}[\mathbf{v}; R]$ is a free resolution of k over R, hence we get

$$\operatorname{Tor}_n^R(k, {}^{\phi^i}\!R) = \mathrm{H}_n(\mathrm{K}[\mathbf{v}; {}^{\phi^i}\!R]) = \mathrm{H}_n(\mathrm{K}[\phi^i(\mathbf{v}); R]).$$

The largest n, such that $\operatorname{Tor}_n^R(k, {}^{\phi^i}\!R) \neq 0$, is equal to $\operatorname{flat\,dim}_R({}^{\phi^i}\!R)$, see (2.2). The largest n, such that $\mathrm{H}_n(\mathrm{K}[\phi^i(\mathbf{v}); R]) \neq 0$, is equal to $\dim R - g$, where g is the maximal length of an R-regular sequence in the ideal $\phi^i(\mathfrak{m})R$, see [26, (16.8)]. The ring R, being regular, is Cohen-Macaulay, so referring to [26, (17.4.i)] we conclude

$$\dim R - g = \dim R - \operatorname{height}(\phi^i(\mathfrak{m})R) = \dim R/(\phi^i(\mathfrak{m})R).$$

The implication (ii) $\Rightarrow$ (iii) is clear. □

Next we show that the complete intersection property of R can also be read off of conditions on a contracting endomorphism ϕ. They are encoded in the growth of the Betti numbers of the module R over ϕ^i. Indeed, the following result is abstracted from Theorem 7.1.5, Theorem 12.1.5, and Corollary 12.2.5.

THEOREM 13.4. *For a contraction $\phi\colon R \to R$ the following are equivalent.*

(i) *R is complete intersection.*

(ii) $\operatorname{cx}_{\phi^i} R \leq \operatorname{codim} R$ *for all integers $i \geq 1$.*

(iii) $\operatorname{cx}_{\phi^i} R < \infty$ *some integer $i \geq 1$.*

(iv) $\operatorname{curv}_{\phi^i} R \leq 1$ *for some integer $i \geq 1$.*

When R has prime characteristic and ϕ is its Frobenius map they are equivalent to

(ii)$'$ $\operatorname{cx}_{\phi^i} R = \operatorname{codim} R$ *for all integers $i \geq 1$.*

As R is regular if and only if $\operatorname{codim} R = 0$, and the flat dimension of ${}^{\phi^i}\!R$ over R is finite if and only if $\operatorname{cx}_{\phi^i} R = 0$, the equivalence of conditions (i), (ii)$'$, and (iii) above constitutes another broad generalization of the theorems of Kunz and Rodicio. The theorem also contains a characterization of complete intersections in terms of Frobenius endomorphisms due to Blanco and Majadas [13, Proposition 1]:

Remark 13.5. Let ϕ be a contraction, such that for some i and some Cohen factorization $\widehat{R} \to R' \to \widehat{R}$ of $\widehat{\phi^i}$ the R'-module $\widehat{R}$ has finite CI-dimension in the sense of [11]. By [11, (5.3)] one then has $\operatorname{cx}_{R'} \widehat{R} < \infty$, so $\operatorname{cx}_{\widehat{\phi^i}} \widehat{R}$ is finite by (7.1.1). Theorem 13.4 now shows that $\widehat{R}$ is complete intersection, and hence so is R.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN, NE 68588
E-mail: avramov@math.unl.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN, NE 68588
E-mail: iyengar@math.unl.edu

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NY 13244
E-mail: cmille06@syr.edu

REFERENCES

[1] M. André, Le caractère additif des déviations des anneaux locaux, *Comment. Math. Helv.* **57** (1982), 648–675.
[2] D. Apassov, Almost finite modules, *Comm. Algebra* **27** (1999), 919–931.
[3] ———, Annihilating complexes of modules, *Math. Scand.* **84** (1999), 11–22.
[4] L. L. Avramov, Homological asymptotics of modules over local rings, *Commutative Algebra (Berkeley, CA, 1987)*, *Math. Sci. Res. Inst. Publ.*, vol. 15, Springer-Verlag, New York, 1989, pp. 33–62.
[5] ———, Modules with extremal resolutions, *Math. Res. Lett.* **3** (1996), 319–328.
[6] ———, Infinite free resolutions, *Six Lectures on Commutative Algebra (Bellaterra, 1996)*, *Progr. Math.*, vol. 166, Birkhäuser, Basel, 1998, pp. 1–118.
[7] ———, Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology, *Ann. of Math.* (2) **150** (1999), 455–487.
[8] L. L. Avramov and H.-B. Foxby, Homological dimensions of unbounded complexes, *J. Pure Appl. Algebra* **71** (1991), 129–155.
[9] ———, Cohen-Macaulay properties of local homomorphisms, *Adv. Math.* **133** (1998), 54–95.
[10] L. L. Avramov, H.-B. Foxby, and B. Herzog, Structure of local homomorphisms, *J. Algebra* **164** (1994), 124–145.
[11] L. L. Avramov, V. N. Gasharov, and I. V. Peeva, Complete intersection dimension, *Inst. Hautes Études Sci. Publ. Math.* **86** (1997), 67–114.
[12] L. L. Avramov and C. Miller, Frobenius powers of complete intersections, *Math. Res. Lett.* **8** (2001), 225–232.
[13] A. Blanco and J. Majadas, Sur les morphismes d'intersection complète en caractéristique p, *J. Algebra* **208** (1998), 35–42.
[14] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar, Finiteness in derived categories of local rings, *Comment. Math. Helv.* (to appear).
[15] D. Ferrand and M. Raynaud, Fibres formelles d'un anneau local noethérien, *Ann. Sci. École Norm. Sup.* **3** (1970), 295–311.
[16] H.-B. Foxby, Bounded complexes of flat modules, *J. Pure Appl. Algebra* **15** (1979), 149–172.
[17] H.-B. Foxby and S. Iyengar, Depth and amplitude of unbounded complexes, *Commutative Algebra. Interactions with Algebraic Geometry (Grenoble–Lyon, 2001)*, *Contemp. Math.*, vol. 331, Amer. Math. Soc., Providence, RI, 2003, pp. 119–137.
[18] R. Hartshorne, *Residues and Duality*, *Lecture Notes in Math.*, vol. 20, Springer-Verlag, Berlin, 1966.
[19] S. Iyengar, Depth for complexes, and intersection theorems, *Math. Z.* **230** (1999), 545–567.
[20] S. Iyengar and S. Sather-Wagstaff, G-dimensions of modules over local homomorphisms. Applications to the Frobenius endomorphism, *Ill. J. Math.* **48** (2004), 241–272.
[21] L. Khatami and S. Yassemi, Gorenstein dimension, Bass formula and Gorenstein rings, preprint, 2003; http://arxiv.org/abs/math.AC/0312513.
[22] J. Koh and K. Lee, Some restrictions on the maps in minimal resolutions, *J. Algebra* **202** (1998), 671–689.
[23] E. Kunz, Characterization of regular local rings of characteristic p, *Amer. J. Math.* **41** (1969), 772–784.
[24] J. Lescot, Séries de Poincaré et modules inertes, *J. Algebra* **132** (1990), 22–49.

[25] G. Levin and W. V. Vasconcelos, Homological dimensions and Macaulay rings, *Pacific J. Math.* **25** (1968), 315–323.

[26] H. Matsumura, *Commutative Ring Theory*, *Cambridge Stud. Adv. Math.*, vol. 8, Cambridge Univ. Press, Cambridge, 1986.

[27] J. McCleary, *A User's Guide to Spectral Sequences*, 2nd ed., *Cambridge Stud. Adv. Math.* vol. 58, Cambridge Univ. Press, Cambridge, 2001.

[28] C. Miller, The Frobenius endomorphism and homological dimensions, *Commutative Algebra. Interactions with Algebraic Geometry (Grenoble–Lyon, 2001)*, *Contemp. Math.*, vol. 331, Amer. Math. Soc., Providence, RI, 2003, pp. 207–234.

[29] C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale, *Inst. Hautes Études Sci. Publ. Math.* **42** (1973), 47–119.

[30] P. Roberts, *Homological Invariants of Modules over Commutative Rings*, *Sem. Math. Sup.*, vol. 72, Presses Univ. Motréal, Montréal, 1980.

[31] ———, *Multiplicities and Chern Classes Local Algebra*, *Cambridge Tracts Math.*, vol. 133, Cambridge Univ. Press, Cambridge, 1998.

[32] A. G. Rodicio, *On a Result of Avramov*, Manuscripta Math. **62** (1988), 181–185.

[33] J.-P. Serre, *Algèbre Locale. Multiplicités*, *Lecture Notes in Math.*, vol. 11, Springer-Verlag, Berlin, 1965.

[34] N. Spaltenstein, Resolutions of unbounded complexes, *Compositio Math.* **65** (1988), 121–154.

[35] R. Takahashi and Y. Yoshino, Characterizing Cohen-Macaulay local rings by Frobenius maps, *Proc. Amer. Math. Soc.* **132** (2004), 3177–3187.

[36] N.-V. Trung, The largest non-vanishing degree of graded local cohomology modules, *J. Algebra* **215** (1999), 481–499.

[37] J.-L. Verdier, Des catégories dérivées des catégories abéliennes, *Astérisque* **239** (1996).